GLOBAL WELLPOSEDNESS IN THE ENERGY SPACE FOR THE MAXWELL-SCHRÖDINGER SYSTEM

IOAN BEJENARU AND DANIEL TATARU

ABSTRACT. We prove that the Maxwell-Schrödinger system in $\mathbb{R}^{3+1}$ is globally well-posed in the energy space. The key element of the proof is to obtain a short time wave packet parametrix for the magnetic Schrödinger equation, which leads to linear, bilinear and trilinear estimates. These, in turn, are extended to larger time scales via a bootstrap argument.

1. INTRODUCTION

The Maxwell-Schrödinger system in $\mathbb{R}^{3+1}$ describes the evolution of a charged non-relativistic quantum mechanical particle interacting with the classical electro-magnetic field it generates. It has the form

$$\begin{cases} iu_t - \Delta_A u = \phi u \\ -\Delta\phi + \partial_t \operatorname{div} A = \rho, & \rho = |u|^2 \\ \Box A + \nabla(\partial_t \phi + \operatorname{div} A) = J, & J = 2Im(\bar{u}, \nabla_A u) \end{cases} \tag{1}$$

where u is the wave function of the particle, (ϕ, A) is the electro-magnetic potential,

$$(u, A, \phi) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \times \mathbb{R}^3$$

and $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$.

The system is invariant under the gauge transform:

$$(u', \phi', A') \to (e^{i\lambda} u, \phi - \partial_t \lambda, A + \nabla\lambda), \qquad \lambda : R^3 \times \mathbb{R} \to \mathbb{R}$$

where $\lambda : R^3 \times \mathbb{R} \to \mathbb{R}$. To remove this degree of freedom we need to fix the gauge. In this article we choose to work in the Coulomb gauge

$$\operatorname{div} A = 0 \tag{2}$$

Under this assumption, the system can be rewritten as:

$$\begin{cases} iu_t - \Delta_A u = \phi u \\ \Box A = PJ \end{cases} \tag{3}$$

where $\phi = (-\Delta)^{-1}(|u|^2)$ and $P = 1 - \nabla \operatorname{div} \Delta^{-1}$ is the projection on the divergence free vectors functions - also called Helmholtz projection. We

The first author was partially supported by NSF grant DMS0738442.
The second author was partially supported by NSF grant DMS0354539.

consider the above system with a set of initial data chosen in Sobolev spaces:

$$(u(0), A(0), A_t(0)) = (u_0, A_0, A_1) \in H^s \times H^\sigma \times H^{\sigma-1}$$

The gauge condition (2) is conserved in time provided the initial data (A_0, A_1) satisfies it due to the form of the second equation in (3).

The conserved quantities associated to the system are the charge and the energy,

$$Q(u) = \int_{\mathbb{R}^3} |u|^2 dx$$

$$E(u) = \int_{\mathbb{R}^3} |\nabla_A u|^2 + \frac{1}{2}(|A_t|^2 + |\nabla_x A|^2) + \frac{1}{2}|\nabla\phi|^2 dx$$

The local well-posedness of the system in various Sobolev spaces above the energy level is known, see [12], [9]. On the other hand the existence of weak energy solutions is established in [2]. The main outstanding problem which we seek to address is the *well-posedness in the energy space*. Our result is

Theorem 1. *The Maxwell-Schrödinger system* (3) *is globally well-posed in the energy space $H^1 \times H^1 \times L^2$ in the following sense:*

i) (regular solutions) For each initial data

$$(u_0, A_0, A_1) \in H^2 \times H^2 \times H^1$$

there exists an unique global solution

$$(u, A) \in C(\mathbb{R}, H^2) \times C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, H^1).$$

ii) (rough solutions) For each initial data

$$(u_0, A_0, A_1) \in H^1 \times H^1 \times L^2$$

there exists a global solution

$$(u, A) \in C(\mathbb{R}, H^1) \times C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2).$$

which is the unique strong limit of the regular solutions in (i).

iii) (continuous dependence) The solutions (u, A) in (ii) depend continuously on the initial data in $H^1 \times H^1 \times L^2$.

We remark that in the process of proving the above results we establish some additional regularity properties for the energy solutions (u, A) which suffice both for the uniqueness and the continuous dependence results. Traditionally these regularity properties are described using $X^{s,b}$ type spaces. Instead here we use the related U^2 and V^2 type spaces associated to both the wave equation and the magnetic Schrödinger equation. These are introduced in the next section; for more details we refer the reader to [7], [6], [3].

Remark 2. *We note that in some directions our analysis yields stronger results than as stated in the theorem. Precisely, the same arguments as those in Section 5 also yield:*

a) Local in time a-priori estimates in $H^\beta \times H^1 \times L^2$ *for* $\beta > \frac{1}{2}$*. This is exactly the range allowed for* β *in Lemma 25 and Lemma 26 (a).*

b) Local in time well-posedness in $H^\beta \times H^1 \times L^2$ *for* $\beta > \frac{3}{4}$*. This reduced range arises due to Lemma 26 (b).*

The nonlinearities on the right hand side of both equations in (3) are fairly mild. Indeed, if Δ_A were replaced by Δ then it would be quite straightforward to iteratively close the argument in $X^{s,b}$ or Strichartz spaces. For the magnetic potential A it is quite reasonable to hope to obtain an $X^{s,b}$ type regularity. Thus the main difficulty stems from the linear magnetic Schrödinger equation

$$iu_t - \Delta_A u = f, \qquad u(0) = u_0 \tag{4}$$

The linear and bilinear estimates for L^2 solutions to (4) are summarized in Theorem 9 in Section 4. The rest of the section is devoted to the well-posedness of (4) in H^2, H^{-2} and intermediate spaces.

The proof of our main result is completed in the following section. The first step is to establish local in time a-priori bounds for solutions to (3), first in H^1 and then in more regular spaces. This is done by treating the nonlinearities on the right of the equations in a perturbative manner. The transition from local in time to global in time is straightforward, using the conserved energy. The second step is to establish the continuous dependence on the initial data. This is a consequence of a Lipschitz dependence result in a weaker topology. Precisely, we show that the corresponding linearized equation is well-posed in $L^2 \times H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$.

The rest of the paper is devoted to the study of L^2 solutions for (4), with the aim of proving Theorem 9. Previous approaches establish Strichartz estimates with a loss of derivatives for this equation in a perturbative manner, starting from the free Schrödinger equation. This no longer suffices for A in the energy space, and instead one needs to study directly the dispersive properties for the linear magnetic Schrödinger equation. Our approach uses some of the ideas described in [5] and [6].

To each dyadic frequency λ, we associate the time scale λ^{-1}. On this time scale we show that at frequency λ the equation (4) is well approximated by its paradifferential truncation, which is roughly

$$iu_t - \Delta_{A_{<\sqrt{\lambda}}} u = f, \qquad u(0) = u_0 \tag{5}$$

Following the ideas in [1], [8], in Sections 6, 7 we obtain a wave packet parametrix for the equation (5) on the λ^{-1} time scale. This allows us to prove sharp Strichartz and square function estimates, as well as bilinear L^2 bounds and trilinear estimates.

In the last section we extend the linear, bilinear and trilinear estimates to larger time scales. A brute force summation of the short time bounds yields unacceptably large constants. Heuristically, the summation can be improved by taking advantage of the localized energy estimates for the magnetic Schrödinger equation. However these are not straightforward. Our

idea is to obtain them from a weaker generalized wave packet decomposition where the localization scales in position and frequency are relaxed as the time scale is iteratively increased.

2. Definitions

Throughout the paper we use the standard Lebesgue spaces L^q_x and mixed space-time versions $L^p_t L^q_x$ which are defined in the standard way. To measure regularity of functions at fixed time we use the standard Sobolev spaces H^s_x. Additional space-time structures will be defined in the next section.

We now introduce dyadic multipliers and a Littlewood-Paley decomposition in frequencies. Throughout the paper the letters λ, μ, ν and γ will be used to denote dyadic values, i.e. $\lambda = 2^i$ for some $i \in \mathbb{N}$.

We say that a function u is localized at frequency λ if its Fourier transform is supported in the annulus $\{|\xi| \in [\frac{\lambda}{8}, 8\lambda]\}$ if $\lambda \geq 2$, respectively in the ball $\{|\xi| \leq 8\}$ if $\lambda = 1$.

By S_λ we denote a multiplier with smooth symbol $s_\lambda(\xi)$ which is supported in the annulus $\{|\xi| \in [\frac{\lambda}{2}, 2\lambda]\}$ for $\lambda \geq 2$ respectively in the ball $\{|\xi| \leq 2\}$ if $\lambda = 1$ and satisfies the bounds

$$|\partial^\alpha s_\lambda(\xi)| \leq c_\alpha \lambda^{-|\alpha|} \tag{6}$$

By $S_{<\lambda}$ we denote a multiplier with smooth symbol $s_{<\lambda}(\xi)$ which is supported in the ball $\{|\xi| \leq 2\lambda\}$, equals 1 in the ball $\{|\xi| \leq \lambda/2\}$ and satisfies (6). All implicit constants in the estimates involving S_λ, $S_{<\lambda}$ will depend on finitely many seminorms of its symbol, i.e. on c_α for $|\alpha| \leq N$, for some large N.

Associated to each λ we also consider $\tilde{S}_\lambda$ to be a multiplier whose symbol $\tilde{s}_\lambda$ satisfies (6), is supported in $\{|\xi| \in [\frac{\lambda}{4}, 4\lambda]\}$ and equals 1 in the support of $\{|\xi| \in [\frac{\lambda}{2}, 2\lambda]\}$. The last condition implies that

$$\tilde{S}_\lambda S_\lambda = S_\lambda$$

Similarly we consider $\tilde{\tilde{S}}_\lambda$ to be a multiplier whose symbol satisfies (6), is supported in $\{|\xi| \in [\lambda/8, 8\lambda]\}$ and equals 1 in the support of $\{|\xi| \in [\lambda/4, 4\lambda]\}$.

3. V^2 and U^2 type spaces

Let H be a Hilbert space. Let $V^2 H$ be the space of right continuous H valued functions on $\mathbb{R}$ with bounded 2-variation:

$$\|u\|^2_{V^2 H} = \sup_{(t_i) \in T} \sum_i \|u(t_{i+1}) - u(t_i)\|^2_H$$

where T is the set of finite increasing sequences in $\mathbb{R}$.

Let $U^2 H$ be the atomic space defined by the atoms:

$$u = \sum_i h_i \chi_{[t_i, t_{i+1})}, \quad \sum_i \|h_i\|^2_H = 1$$

for some $(t_i) \in T$. We have the inclusion $U^2H \subset V^2H$ but in effect these spaces are very close, and also close to the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}}$. Precisely, we can bracket them using homogeneous Besov spaces as follows:

$$\dot{B}^{\frac{1}{2}}_{2,1} \subset U^2 \subset V^2 \subset \dot{B}^{\frac{1}{2}}_{2,\infty} \tag{7}$$

We denote by DU^2H the space of (distributional) derivatives of U^2H functions. Then there is also a duality relation between V^2H and U^2H, namely

$$(DU^2H)^* = V^2H \tag{8}$$

For more details on the U^2 and V^2 spaces we refer the reader to [7] and [3].

Given an abstract evolution in H,

$$iu_t = B(t)u, \qquad u(0) = u_0$$

which generates a family of bounded evolution operators

$$S(t,s) : H \to H, \qquad t, s \in \mathbb{R}$$

we can define the associated spaces U^2_BH, V^2_BH and DU^2_BH by

$$\|u\|_{V^2_BH} = \|S(0,t)u(t)\|_{V^2H}, \qquad \|u\|_{U^2_BH} = \|S(0,t)u(t)\|_{V^2H}$$

respectively

$$DU^2_BL^2 = \{(i\partial_t - B)u;\ u \in U^2_BH\}$$

On occasion we need to compare the above spaces associated to closely related operators. For this we use the following

Lemma 3. *Let H be a Hilbert space and $B(t)$, $C(t)$ two families of bounded selfadjoint operators in H. Suppose that for ε sufficiently small we have*

$$\|(B-C)u\|_{DU^2_BH} \leq \varepsilon \|u\|_{U^2_BH}$$

Then

$$\|u\|_{U^2_BH} \approx \|u\|_{U^2_CH}, \qquad \|u\|_{V^2_BH} \approx \|u\|_{V^2_CH}, \qquad \|f\|_{DU^2_BH} \approx \|f\|_{DU^2_CH}$$

Proof. By conjugating with respect the B flow we can assume without any restriction in generality that $B = 0$. Then we can solve the equation

$$iu_t - Cu = 0, \qquad u(0) = u_0$$

by treating C perturbatively to obtain a solution

$$u = u_0 + \epsilon u_1(t), \qquad \|u_1\|_{U^2H} \lesssim \|u_0\|_H$$

Applying this to each step in U^2_C atoms we obtain

$$\|u\|_{U^2H} \lesssim \|u\|_{U^2_CH}$$

for arbitrary u.

For the converse, applying the above result to each step in a U^2H atom we conclude that for each $u \in U^2H$ we can find $u_1 \in U^2H$ so that

$$\|u + \epsilon u_1\|_{U^2_C H} + \|u_1\|_{U^2 H} \lesssim \|u\|_{U^2 H}$$

Iterating this shows that

$$\|u\|_{U^2_C H} \lesssim \|u\|_{U^2 H}$$

Hence $U^2_C H = U^2 H$.

Consider now $f \in DU^2_C H$. Then $f = iu_t - Cu$ for some $u \in U^2_C H = U^2 H$. Since C maps $U^2 H$ to $DU^2 H$ this implies that $f \in DU^2 H$. Conversely, if $f \in DU^2 H$ then we can solve the inhomogeneous equation

$$iu_t - Cu = f, \qquad u(0) = 0$$

iteratively in $U^2 H$. This gives a solution $u \in U^2 H = U^2_C H$, therefore $f \in DU^2_C H$. We have proved that $DU^2 H = DU^2_C H$. Then the last relation $V^2 H = V^2_C H$ follows by duality. □

Following the above procedure we can associate similar spaces to the Schrödinger flow by pulling back functions to time 0 along the flow, namely

$$\|u\|_{V^2_\Delta L^2} = \|e^{it\Delta} u\|_{V^2 L^2}, \qquad \|u\|_{U^2_\Delta L^2} = \|e^{it\Delta} u\|_{V^2 L^2}$$

The magnetic Schrödinger equation has time dependent coefficients, so we replace the above exponential with the corresponding evolution operators. We denote by $S^A(t,s)$ the family of evolution operators corresponding to the equation (4). These are L^2 isometries. Then we define

$$\|u\|_{V^2_A L^2} = \|S(0,t)u(t)\|_{V^2 L^2}, \qquad \|u\|_{U^2_A L^2} = \|S(0,t)u(t)\|_{V^2 L^2}$$

These spaces turn out to be a good replacement for the $X^{0,\frac{1}{2}}$ space associated to the Schrödinger equations. We also define

$$DU^2_A L^2 = \{(i\partial_t - \Delta_A)u;\ u \in U^2_A L^2\}$$

By (8) we have the duality relation

$$(DU^2_A L^2)^* = V^2_A L^2$$

When solving the equation (4) we let $f \in DU^2_A L^2$, and we have the straightforward bound

$$\|u\|_{U^2_A L^2} \lesssim \|u_0\|_{L^2} + \|f\|_{DU^2_A L^2} \tag{9}$$

In our study of nonlinear equations later on we need to estimate multilinear expressions in $DU^2_A L^2$. By duality, this is always turned into multilinear estimates involving $V^2_A L^2$ functions.

Finally, we define similar spaces associated to the wave equation. The wave equation is second order in time therefore we use a half-wave decomposition and set

$$\|u\|_{V^2_\pm L^2} = \|e^{\pm it|D|} u\|_{V^2 L^2}, \qquad \|u\|_{U^2_\pm L^2} = \|e^{\pm it|D|} u\|_{V^2 L^2}$$

Then the spaces for the full wave equation are

$$\|u\|_{U^2_W L^2} = \|u\|_{U^2_+ L^2 + U^2_- L^2}, \qquad \|u\|_{V^2_W L^2} = \|u\|_{V^2_+ L^2 + V^2_- L^2}$$

For the inhomogeneous term in the wave equation we use the space $DU^2_W L^2$ with norm

$$\|f\|_{DU^2_W L^2} = \|f\|_{DU^2_+ L^2 \cap DU^2_- L^2}$$

Then to solve the inhomogeneous wave equation we use

$$\|\nabla u\|_{U^2_W L^2} \lesssim \|\nabla u(0)\|_{L^2} + \|\Box u\|_{DU^2_W L^2}$$

Similarly we set

$$\|u\|_{U^2_W H^s} = \|\langle D_x\rangle^s u\|_{U^2_+ L^2 + U^2_- L^2}, \quad \|u\|_{V^2_W H^s} = \|\langle D_x\rangle^s u\|_{V^2_+ L^2 + V^2_- L^2}$$

and

$$\|f\|_{DU^2_W H^s} = \|\langle D_x\rangle^s f\|_{DU^2 L^2}$$

Such spaces originate in unpublished work of the second author on the wave-map equation, and have been successfully used in various contexts so far, see [7], [6],[1],[3]. The Strichartz estimates for the wave equation turn into embeddings for $U^2_W H^s$ spaces. If the indices (p,q) satisfy

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 2 < p \le \infty \tag{10}$$

then we have

$$\|u\|_{L^p L^q} \lesssim \|u\|_{V^2_W H^{\frac{2}{p}}} \tag{11}$$

If we consider frequency localized solutions to the wave equation on a very small time scale, then the wave equation is ineffective. Precisely,

Lemma 4. *Let $B_{<\lambda}$ be a function which is localized at frequency $< \lambda$. Then*

$$\|B_{<\lambda}\|_{U^2_W(I;L^2)} \approx \|B_{<\lambda}\|_{U^2(I;L^2)}, \qquad |I| \le \lambda^{-1} \tag{12}$$

Proof. This follows from the similar bound for the corresponding half-wave spaces $U^2_\pm L^2$, and by Lemma 3 it is a consequence of the fact that for a short time the spatial derivatives in the half wave equation can be treated perturbatively,

$$\begin{aligned} \||D_x|B_{<\lambda}\|_{DU^2(I,L^2)} &\lesssim \||D_x|B_{<\lambda}\|_{L^1(I,L^2)} \lesssim \lambda\|B_{<\lambda}\|_{L^1(I,L^2)} \\ &\lesssim |I|\lambda\, \|B_{<\lambda}\|_{L^\infty(I,L^2)} \lesssim |I|\lambda\, \|B_{<\lambda}\|_{U^2(I,L^2)} \end{aligned}$$

□

The finite speed of propagation for the wave equation allows us to spatially localize functions in the $U^2_W L^2$ spaces. For $R > 0$ we consider a covering $(Q_i^R)_{i\in\mathbb{Z}^3}$ of $\mathbb{R}^3$ with cubes of size R. Let χ_i^R be an associated smooth partition of unity. Then we have the following result:

Lemma 5. *Assume that I is a time interval with $|I| \le R$. Then:*

$$\sum_{i\in\mathbb{Z}^3} \|\chi_i^R u\|^2_{U^2_W(I;L^2)} \lesssim \|u\|^2_{U^2_W(I;L^2)} \tag{13}$$

Proof. By rescaling we can take $R = 1$. Without any restriction in generality we can also assume that $|I| = 1$. We prove that the result holds for one of the two half-wave spaces, say $U^2_+(I;L^2)$. It is enough to verify (13) for atoms, and further for each step in an atom. Hence we can assume that u solves the half wave equation

$$(i\partial_t + |D_x|)u = 0$$

Then we have

$$(i\partial_t + |D_x|)(\chi_i^R u) = [|D_x|, \chi_i^R]u$$

By standard commutator estimates we have at fixed time

$$\sum_{i\in\mathbb{Z}^3} \|[|D_x|, \chi_i^R]u(t)\|^2_{L^2} \lesssim \|u(t)\|^2_{L^2}$$

Then

$$\sum_{i\in\mathbb{Z}^3} \|\chi_i^R u\|^2_{U^2_W(I;L^2)} \lesssim \sum_{i\in\mathbb{Z}^3} \|\chi_i^R u(0)\|^2_{L^2} + \|[|D_x|, \chi_i^R]u(t)\|^2_{L^1L^2} \lesssim \|u(0)\|^2_{L^2}$$

□

Due to the atomic structure, in many estimates it is convenient to work with the U^2 type spaces instead of V^2. In order to transfer the estimates from U^2 to V^2 we use the following result from [3]:

Proposition 6. *Let $2 < p < \infty$. If $u \in V^2H$ then for each $0 < \varepsilon < 1$ there exist $u_1 \in U^2H$ and $u_2 \in U^pH$ such that $u = u_1 + u_2$ and*

$$|\ln\varepsilon|^{-1}\|u_1\|_{U^2H} + \varepsilon^{-1}\|u_2\|_{U^pH} \lesssim \|u\|_{V^2H} \tag{14}$$

Here U^p is defined in the same manner as U^2 but with the l^2 summation replaced by an l^p summation. One way we use this result is as follows:

Corollary 7. *Let $\varepsilon > 0$ and N arbitrarily large. Then*

$$V^2_W H^s \subset U^2_W H^{s-\epsilon} + L^\infty H^N$$

Following is another example of how this result can be applied. Typically in our analysis we prove dyadic trilinear estimates of the form

$$\left| \int_I \int_{\mathbb{R}}^3 S_{\lambda_1} u S_{\lambda_2}\bar{v}, S_{\lambda_3} B dx dt \right| \lesssim C_1(|I|, \lambda_{123}) \|u\|_{U^2_A L^2} \|v\|_{U^2_A L^2} \|B\|_{U^2_W L^2} \tag{15}$$

What we need instead is an estimate with one U^2 replaced by a V^2, say

$$\left| \int_I \int_{\mathbb{R}}^3 S_{\lambda_1} u S_{\lambda_2}\bar{v}, S_{\lambda_3} B dx dt \right| \lesssim C_2(|I|, \lambda_{123}) \|u\|_{U^2_A L^2} \|v\|_{U^2_A L^2} \|B\|_{V^2_W L^2} \tag{16}$$

Denoting

$$\lambda_{max} = \max\{\lambda_1, \lambda_2, \lambda_3\},$$

due to Proposition 6 we can easily show that

Lemma 8. *Assume* (15) *holds and* $|I| \leq 1$. *Then* (16) *holds with*

$$C_2(|I|, \lambda_{123}) = C_1(|I|, \lambda_{123}) \ln \lambda_{max}.$$

The same holds if the V^2 *structure is placed on any of the other two factors in* (16).

Proof. Without any restriction in generality we assume that λ_1, λ_2 and λ_3 are so that the integral in (15) is nontrivial. Taking u, v and B to be time independent frequency localized bump functions we easily see that

$$C_1(|I|, \lambda_{123}) \gtrsim |I| \lambda_{max}^{-N} \tag{17}$$

for some sufficiently large N.

For each $0 < \epsilon \leq 1$ we decompose $B = B_1 + B_2$ as in Proposition 6. For B_1 we use (15) while for B_2 we use Bernstein's inequality to estimate

$$\begin{aligned}\left| \int_I \int_{\mathbb{R}}^3 S_{\lambda_1} u S_{\lambda_2} \bar{v}, S_{\lambda_3} B_2 dx dt \right| &\lesssim |I| \lambda_{max}^N \|u\|_{L^\infty L^2} \|v\|_{L^\infty L^2} \|B_2\|_{L^\infty L^2} \\ &\lesssim |I| \lambda_{max}^N \|u\|_{U_A^2 L^2} \|v\|_{U_A^2 L^2} \|B_2\|_{U_W^p L^2}\end{aligned}$$

Adding the B_1 and the B_2 bounds gives

$$C_2(|I|, \lambda_{123}) \lesssim |\ln \epsilon| C_1(|I|, \lambda_{123}) + \epsilon |I| \lambda_{max}^N$$

We set $\epsilon = \lambda_{max}^{-2N}$. Then the conclusion of the Lemma follows due to (17). □

4. The linear magnetic Schrödinger equation

In this section we summarize the key properties of solutions to the homogeneous and inhomogeneous linear magnetic Schrödinger equation

$$iu_t - \Delta_A u = 0, \qquad u(0) = u_0 \tag{18}$$

$$iu_t - \Delta_A u = f, \qquad u(0) = u_0 \tag{19}$$

in L^2, and we use them in order to show that the above equation is also well-posed in H^2, H^{-2} and in intermediate spaces.

We assume that $A \in U_W^2 H^1$ with $\nabla \cdot A = 0$. All constants in the estimates depend on the $U_W^2 H^1$ norm of A which is why we introduce the notation $X \lesssim_A Y$, which means $X \leq C(\|A\|_{U_W^2 H^1}) Y$.

The trilinear estimates are concerned with integrals of the form

$$I_{\lambda_1, \lambda_2, \lambda_3}^T(u, v, B) = \int_0^T \int_{\mathbb{R}^3} S_{\lambda_1} u \, S_{\lambda_2} \bar{v} \, S_{\lambda_3} B \; dx dt$$

where u and v are associated to the magnetic Schrödinger equation and B is associated to the wave equation. In order for the above integral to be

nontrivial the two highest frequencies need to be comparable. Thus by a slight abuse of notation in the sequel we restrict ourselves to the case

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{\lambda, \lambda, \mu\}, \qquad \mu \leq \lambda$$

With these notations we have

Theorem 9. *For each $A \in U^2_W H^1$ with $\nabla \cdot A = 0$ the equation (18) is well-posed in L^2. For each $\epsilon > 0$ there exists $\delta > 0$ so that the following properties hold for $0 < T \leq 1$:*

(i) Strichartz estimates:

$$\|S_\lambda u\|_{L^p(0,T;L^q)} \lesssim_A T^{\frac{\delta}{p}} \lambda^{\frac{1}{p}} \|u\|_{U^2_A L^2} \qquad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq p \leq \infty \tag{20}$$

(ii) Local energy estimates. For any spatial cube Q of size 1 we have

$$\|S_\lambda u\|_{L^2(0,T;Q)} \lesssim_A T^\delta \lambda^{-\frac{1}{2}+\varepsilon} \|u\|_{U^2_A L^2}. \tag{21}$$

(iii) Local Strichartz estimates. For any spatial cube Q of size 1 we have:

$$\|S_\lambda u\|_{L^2(0,T;L^6(Q))} \lesssim_A T^\delta \lambda^\epsilon \|u\|_{U^2_A L^2}. \tag{22}$$

(iv) Trilinear estimates. For any $0 < T \leq 1$ and $\mu \leq \lambda$ we have

$$|I^T_{\lambda,\lambda,\mu}(u, v, B)| \lesssim_A T^\delta \lambda^\epsilon \min(1, \mu\lambda^{-\frac{1}{2}}) \|u\|_{U^2_A L^2} \|v\|_{U^2_A L^2} \|B\|_{U^2_W L^2}. \tag{23}$$

On the other hand if $\mu \ll \lambda$ then

$$|I^T_{\lambda,\mu,\lambda}(u, v, B)| \lesssim_A T^\delta \lambda^{-\frac{1}{2}+\epsilon} \mu^{\frac{1}{2}} \|u\|_{U^2_A L^2} \|v\|_{U^2_A L^2} \|B\|_{U^2_W L^2} \tag{24}$$

The proof of this theorem is quite involved and is relegated to Sections 6-10.

The smallness given by the T^δ factor is needed in several proofs which use either the contraction principle or bootstrap arguments. However, this factor is nontrivial only in (20). Indeed, we have

Remark 10. *Assume that the conclusion of Theorem 9 holds without the T^δ factor in (21), (22), (23) and (24). Then the conclusion of Theorem 9 holds in full.*

Proof. For (21) we observe that

$$\|S_\lambda u\|_{L^2(0,T;Q)} \lesssim T^{\frac{1}{2}} \|u\|_{L^\infty L^2} \lesssim \|u\|_{U^2_A L^2}.$$

Interpolating this with (21) without the T^δ factor yields (21) with a T^δ factor. By Bernstein's inequality the same argument works for (22).

For (23) we can also write the obvious estimate

$$|I^T_{\lambda,\lambda,\mu}(u, v, B)| \lesssim_A T\mu^{\frac{3}{2}} \|u\|_{L^\infty L^2} \|v\|_{L^\infty L^2} \|B\|_{L^\infty L^2}.$$

which is then interpolated with (23) without the T^δ factor. The same argument applies for (24). $\square$

Next we turn our attention to the H^2 and H^{-2} well-posedness for (18). We make the transition from L^2 to H^2 and H^{-2} using the coercive elliptic operator $1-\Delta_A$. Its properties are summarized in the following

Lemma 11. *For each $0 \leq s \leq 2$ the operator $1-\Delta_A$ is a diffeomorphism*

$$1-\Delta_A : H^s \to H^{s-2}$$

which depends continuously on $A \in H^1$.

The proof uses standard elliptic arguments and is left for the reader.

Using the above operator we define the spaces $U^2_A H^2$, $V^2_A H^2$, respectively $DU^2_A H^2$ by

$$\|u\|_{\tilde{U}^2_A H^2} = \|(1-\Delta_A)u\|_{U^2_A L^2}, \qquad \|u\|_{\tilde{V}^2_A H^2} = \|(1-\Delta_A)u\|_{V^2_A L^2}$$

respectively

$$\|f\|_{D\tilde{U}^2_A H^2} = \|(1-\Delta_A)f\|_{DU^2_A L^2}$$

Remark 12. *The reason we use the $\tilde{U}$, $\tilde{V}$ notation above is to differentiate these spaces from the $U^2_A H^2$, $V^2_A H^2$, $DU^2_A H^2$ spaces which should be defined as in the previous section, with respect to the H^2 flow of* (18)*. This is not possible at this point, as we have not yet proved that* (18) *is well-posed in H^2. However, after we do so we will prove that the above two sets of norms are equivalent. After that the $\tilde{U}$, $\tilde{V}$ notation is dropped.*

We can transfer the estimates from Theorem 9 to the $U^2_A H^2$ spaces by making an elliptic transition between Δ_A and Δ:

Lemma 13. *Let $A \in U^2_W H^1$ with $\nabla \cdot A = 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ so that the following properties hold:*

(i) For p, q as in (20) *we have the Strichartz estimate*

$$\|S_\lambda u\|_{L^p(0,T;L^q)} \lesssim_A T^{\frac{\delta}{p}} \lambda^{-2+\frac{1}{p}} \|u\|_{\tilde{U}^2_A H^2} \tag{25}$$

(ii) Elliptic representation. Each $u \in U^2_A H^2$ can be expressed as

$$u = (1-\Delta)^{-1}(u_e + u_r), \qquad \|u_e\|_{U^2_A L^2} + T^{-\delta}\|u_r\|_{L^2(0,T;H^{1-\epsilon})} \lesssim_A \|u\|_{\tilde{U}^2_A H^2} \tag{26}$$

(iii) Local energy estimates. For any spatial cube Q of size 1 *we have:*

$$\|S_\lambda u\|_{L^2([0,T]\times Q)} \lesssim_A T^\delta \lambda^{-2-\frac{1}{2}+\epsilon} \|u_0\|_{\tilde{U}^2_A H^2} \tag{27}$$

(iv) Local Strichartz estimates. For any spatial cube Q of size 1 *we have:*

$$\|S_\lambda u\|_{L^2(0,T;L^6(Q))} \lesssim_A T^\delta \lambda^{-2+\epsilon} \|u_0\|_{\tilde{U}^2_A H^2} \tag{28}$$

Proof. (i) The Strichartz estimate (25) follows from

$$\|S_\lambda (1-\Delta) u\|_{L^p L^q} \lesssim_A T^{\frac{\delta}{p}} \lambda^{\frac{1}{p}} \|u\|_{\tilde{U}^2_A H^2} \tag{29}$$

We use the identity

$$(1-\Delta)u = (1-\Delta_A)u - 2iA\nabla u - A^2 u = (1-\Delta_A)u - R_A(u) \tag{30}$$

and estimate each of the three terms. From the definition of $\tilde{U}^2_A H^2$ and (20) we have:

$$\|S_\lambda(1-\Delta_A)u\|_{L^pL^q} \lesssim_A T^{\frac{\delta}{p}}\lambda^{\frac{1}{p}}\|u\|_{\tilde{U}^2_A H^2}$$

For the second term we use Bernstein's inequality and the exponent relation in (20) to estimate

$$\begin{aligned}\|S_\lambda(A\nabla u)\|_{L^pL^q} &\lesssim_A T^{\frac{1}{p}}\lambda^{\frac{2}{p}}\|S_\lambda(A\nabla u)\|_{L^\infty L^2}\\ &\lesssim T^{\frac{1}{p}}\lambda^{\frac{1}{p}}\|A\nabla u\|_{L^\infty H^{\frac{1}{2}}}\\ &\lesssim T^{\frac{1}{p}}\lambda^{\frac{1}{p}}\|A\|_{L^\infty H^1}\|u\|_{L^\infty H^2}\end{aligned}$$

Similarly for the last term we obtain

$$\|S_\lambda(A^2u)\|_{L^pL^q} \lesssim T^{\frac{1}{p}}\lambda^{\frac{1}{p}}\|A\|^2_{L^\infty H^1}\|u\|_{L^\infty H^2}$$

This concludes the proof for (29) which implies (25).

(ii) By (30) we can set

$$u_e = (1-\Delta_A)u, \qquad u_r = -R_A(u) = -2iA\nabla u - A^2 u$$

Hence it remains to prove that

$$\|S_\lambda(A\nabla u)\|_{L^2} + \|S_\lambda(A^2u)\|_{L^2} \lesssim_A T^\delta \lambda^{-1+\epsilon}\|u\|_{\tilde{U}^2_A H^2} \tag{31}$$

By the argument in Remark 10, here and for the rest of the proof of the lemma we can neglect the T^δ factors.

We decompose the expression $S_\lambda(A\nabla u)$ as

$$S_\lambda(A\nabla u) = S_\lambda\left(\sum_{\gamma\lesssim\lambda} S_\gamma A S_\lambda \nabla u + \sum_{\gamma\lesssim\lambda} S_\lambda A S_\gamma\nabla u + \sum_{\gamma\gtrsim\lambda} S_\gamma A S_\gamma \nabla u\right) \tag{32}$$

For exponents (p,q) satisfying (10) we invoke the Strichartz estimates (11) for the wave equation. By Bernstein's inequality and (25) we can derive a similar bound for u,

$$\|S_\gamma \nabla u\|_{L^qL^p} \lesssim_A \gamma^{-\frac{2}{p}}\|u\|_{\tilde{U}^2_A H^2}. \tag{33}$$

The L^2 bound for the product is obtained by multiplying the last two inequalities. For the first term in (32) we take q close to ∞, for the second we take $p=\infty$, while for the third any choice will do.

For the expression $S_\lambda(A^2u)$ we take a triple Littlewood-Paley decomposition,

$$S_\lambda(A^2u) = \sum_{\lambda_1,\lambda_2,\lambda_3} S_\lambda(A_{\lambda_1}A_{\lambda_2}u_{\lambda_3})$$

Then we must have $\lambda_{max} \geq \lambda$.

If $\lambda_{max} = \lambda_3$ then we use the above Strichartz inequalities and Bernstein's inequality to estimate the triple product as

$$\begin{aligned} \|A_{\lambda_1} A_{\lambda_2} u_{\lambda_3}\|_{L^2} &\lesssim \|A_{\lambda_1}\|_{L^4L^\infty} \|A_{\lambda_2}\|_{L^4L^\infty} \|u_{\lambda_3}\|_{L^\infty L^2} \\ &\lesssim_A \lambda_1^{\frac14} \lambda_2^{\frac14} \lambda_3^{-2} \|A_{\lambda_1}\|_{U^2_W H^1} \|A_{\lambda_2}\|_{U^2_W H^1} \|u_{\lambda_3}\|_{\tilde U^2_A H^2} \end{aligned}$$

The summation with respect to $\lambda_1, \lambda_2, \lambda_3$ is straightforward.

If $\lambda_3 \ll \lambda_{max}$ then there are two possibilities. One is $\lambda_{max} = \lambda$, in which case we assume w.a.r.g. that $\lambda = \lambda_1 \geq \lambda_2, \lambda_3$ and estimate

$$\begin{aligned} \|A_{\lambda_1} A_{\lambda_2} u_{\lambda_3}\|_{L^2} &\lesssim \|A_\lambda\|_{L^pL^q} \|A_{\lambda_2}\|_{L^qL^p} \|u_{\lambda_3}\|_{L^\infty} \\ &\lesssim_A \lambda_1^{-\frac2p} \lambda_2^{-\frac2q} \lambda_3^{-\frac12} \|A_{\lambda_1}\|_{U^2_W H^1} \|A_{\lambda_2}\|_{U^2_W H^1} \|u_{\lambda_3}\|_{\tilde U^2_A H^2} \end{aligned}$$

with q close to infinity. The other possibility is $\lambda_{max} \gg \lambda$, in which case we must have $\lambda_1 = \lambda_2 \gg \lambda_3$. Then we estimate the triple product as above, but the choice of p and q is no longer important. The proof of (31) is concluded.

(iii) We use the representation in (26). The bound for u_r holds without any localization. For u_e we can write

$$S_\lambda (1-\Delta)^{-1} u_e = \lambda^{-2} (\lambda^2 (1-\Delta)^{-1} \tilde S_\lambda) S_\lambda u_e$$

where the symbol of $\tilde S_\lambda$ is still supported at frequency λ but equals 1 in the support of the symbol of S_λ. The operator $\lambda^2 \Delta^{-1} \tilde S_\lambda$ is a unit mollifier acting on the λ^{-1} scale, therefore it is bounded in $l^\infty_Q L^2([0,1] \times Q)$.

(iv) We use the representation in (26). By Bernstein's inequality the bound for u_r holds without any localization. For u_e we argue as above. □

Next we define similar spaces $\tilde U^2_A H^{-2}$, $\tilde V^2_A H^{-2}$, respectively $D\tilde U^2_A H^{-2}$ in a manner similar to the H^2 case, namely

$$\|u\|_{\tilde U^2_A H^{-2}} = \|(1-\Delta_A)^{-1} u\|_{U^2_A L^2}, \qquad |u\|_{\tilde V^2_A H^{-2}} = \|(1-\Delta_A)^{-1} u\|_{V^2_A L^2},$$

respectively

$$\|f\|_{D\tilde U^2_A H^{-2}} = \|(1-\Delta_A)^{-1} f\|_{DU^2_A L^2}$$

Due to the duality relation 8 we have the $H^2 - H^{-2}$ duality

$$(D\tilde U^2_A H^{-2})^* = \tilde V^2_A H^2, \qquad (D\tilde U^2_A H^2)^* = \tilde V^2_A H^{-2} \tag{34}$$

For functions in $U^2_A H^{-2}$ we can prove results similar to Lemma 13:

Lemma 14. *Let $A \in U^2_W H^1$ with $\nabla \cdot A = 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ so that the following properties hold:*

(i) For p, q as in (20) we have the Strichartz estimate

$$\|S_\lambda u\|_{L^p(0,T;L^q)} \lesssim_A T^{\frac{\delta}{p}} \lambda^{2+\frac1p} \|u\|_{\tilde U^2_A H^{-2}} \tag{35}$$

(ii) Elliptic representation. Each $u \in \tilde U^2_A H^2$ can be expressed as

$$u = (1-\Delta)(u_e + u_r), \qquad \|u_e\|_{U^2_A L^2} + T^{-\delta} \|u_r\|_{L^2(0,T;H^{1-\epsilon})} \lesssim_A \|u\|_{\tilde U^2_A H^{-2}} \tag{36}$$

(iii) Local energy estimates. For any spatial cube Q of size 1 we have:

$$\|S_\lambda u\|_{L^2([0,T]\times Q)} \lesssim_A T^\delta \lambda^{2-\frac{1}{2}+\epsilon}\|u_0\|_{\tilde{U}^2_A H^{-2}} \tag{37}$$

(iv) Local Strichartz estimates. For any spatial cube Q of size 1 we have:

$$\|S_\lambda u\|_{L^2(0,T;L^6(Q))} \lesssim_A T^\delta \lambda^{2+\epsilon}\|u_0\|_{\tilde{U}^2_A H^{-2}} \tag{38}$$

Proof. The proof is similar to the proof of Lemma 13, so we merely outline it. We denote $v = (1-\Delta_A)^{-1}u$. Then

$$u = (1-\Delta)v - R_A(v)$$

To prove (35) we use (20) for $(1-\Delta)v$ and it remains to show that

$$\|S_\lambda \nabla(Av)\|_{L^pL^q} + \|S_\lambda(A^2 v)\|_{L^pL^q} \lesssim_A T^\delta \lambda^{2+\frac{1}{p}}\|v\|_{U^2_A L^2}$$

which is obtained using only the energy estimates for A and v.

For (36) we set

$$u_e = v, \qquad u_r = (1-\Delta)^{-1}R_A(v)$$

Then it remains to show that

$$\|S_\lambda \nabla(Av)\|_{L^2} + \|S_\lambda(A^2 v)\|_{L^2} \lesssim_A T^\delta \lambda^{1+\varepsilon}\|v\|_{U^2_A L^2}$$

But this is obtained in the same manner as (31) from the Strichartz estimates for A and u_2.

Finally, (37) and (38) are proved exactly as in the previous lemma. $\square$

Next we turn our attention to the trilinear bounds, namely the H^2 and H^{-2} counterparts of (23) and (24). For uniformity in notations we set $\tilde{U}^2_A L^2 = U^2_A L^2$. Then

Lemma 15. *Let $k, l \in \{-2, 0, 2\}$. Then for any $0 < T \leq 1$ and $\mu \leq \lambda$ we have*

$$|I^T_{\lambda,\lambda,\mu}(u,v,B)| \lesssim_A T^\delta \lambda^\epsilon \min(1, \mu\lambda^{-\frac{1}{2}})\lambda^{-k-l}\|u\|_{\tilde{U}^2_A H^k}\|v\|_{\tilde{U}^2_A H^l}\|B\|_{U^2_W L^2} \tag{39}$$

while if $\mu \ll \lambda$ then

$$|I^T_{\lambda,\mu,\lambda}(u,v,B)| \lesssim_A T^\delta \lambda^{-\frac{1}{2}+\epsilon}\mu^{\frac{1}{2}}\mu^{-l}\lambda^{-k}\|u\|_{\tilde{U}^2_A H^k}\|v\|_{\tilde{U}^2_A H^l}\|B\|_{U^2_W L^2} \tag{40}$$

Proof. The T^δ factor can be neglected by the argument in Remark 10. We represent

$$u = (1-\Delta)^{-\frac{k}{2}}(u_e + u_r)$$

where u_e, u_r are chosen as in (26) if $k = 2$, as in (36) if $k = -2$ and with $u_r = 0$ if $k = 0$. Similarly we set

$$v = (1-\Delta)^{-\frac{k}{2}}(v_e + v_r)$$

We begin with (39) and consider all four combinations. The estimate for u_e and v_e is exactly (23). The estimate for u_e and v_r reads

$$|I^1_{\lambda,\lambda,\mu}(u_e, v_r, B)| \lesssim_A \lambda^\epsilon \min(1, \mu\lambda^{-\frac{1}{2}})\|u_e\|_{U^2_A L^2}\|v_r\|_{L^2 H^{1-\epsilon}}\|B\|_{U^2_W L^2}$$

and is a consequence of the stronger bilinear L^2 estimate

$$\|S_\lambda u_e S_\mu B\|_{L^2} \lesssim_A \mu^{\frac{1}{2}} \lambda^\epsilon \|u_e\|_{U^2_A L^2} \|B\|_{U^2_W L^2} \tag{41}$$

Due to the finite speed of propagation for the wave equation, see (13), we can localize this to the unit spatial scale. But on a unit cube Q we use the local Strichartz estimate (22) for u_e and the energy estimate for B.

The estimate for u_r and v_e is similar. The estimate for u_r and v_r reads

$$|I^1_{\lambda,\lambda,\mu}(u_r, v_r, B)| \lesssim_A \lambda^\epsilon \min(1, \mu\lambda^{-\frac{1}{2}}) \|u_r\|_{L^2 H^{1-\epsilon}} \|v_r\|_{L^2 H^{1-\epsilon}} \|B\|_{U^2_W L^2}$$

and is easily proved using L^2 bounds for $S_\lambda u_r$, $S_\lambda v_r$ and an L^∞ bound for $S_\mu B$.

The proof of (40) is similar. For later use we note the bilinear L^2 estimates, namely

$$\|S_\mu v_e S_\lambda B\|_{L^2} \lesssim_A \mu^{\frac{1}{2}+\epsilon} \|u_e\|_{U^2_A L^2} \|B\|_{U^2_W L^2} \tag{42}$$

respectively

$$\|S_\mu(S_\lambda u_e S_\lambda B)\|_{L^2} \lesssim_A \mu^{\frac{1}{2}+\epsilon} \|u_e\|_{U^2_A L^2} \|B\|_{U^2_W L^2} \tag{43}$$

Both are proved by localizing to a unit spatial scale and then by combining the local Strichartz estimate (22) for u_e and v_e and the energy estimate (11) for B.

□

Now we consider the H^2 well-posedness of (18).

Proposition 16. *Let $A \in U^2_W$ with $\nabla \cdot A = 0$. Then the equation (19) is well-posed in H^2. In addition we have*

$$U^2_A H^2 = \tilde{U}^2_A H^2, \qquad V^2_A H^2 = \tilde{V}^2_A H^2, \qquad DU^2_A H^2 = \tilde{U}^2_A H^2$$

with equivalent norms.

Proof. In order to solve the equation (18) with initial data $u_0 \in H^2$ we consider the equation for $v = (1 - \Delta_A)u$ which has the form

$$(i\partial_t - \Delta_A)v = 2(A_t \nabla - iAA_t)u$$

Expressing u in terms of v we obtain

$$(i\partial_t - \Delta_A)v = 2(A_t \nabla - iAA_t)(1 - \Delta_A)^{-1} v \tag{44}$$

We seek to solve this equation perturbatively in $U^2_A L^2$. For this we need first to establish suitable mapping properties for the operator $A_t \nabla - iAA_t$.

Lemma 17. *The operator $A_t \nabla - iAA_t$ satisfies the space-time bound*

$$\|(A_t \nabla - 2iAA_t)u\|_{DU^2_A L^2} \lesssim_A T^\delta \|u\|_{\tilde{U}^2_A H^2} \tag{45}$$

Proof. By duality, (45) follows from the bounds

$$\left| \int_0^T \int_{\mathbb{R}^3} B \nabla u \bar{v} dx dt \right| \lesssim_A T^{\delta} \|u\|_{\tilde{U}_A^2 H^2} \|v\|_{V_A^2 L^2} \|B\|_{U_W^2 L^2} \tag{46}$$

respectively

$$\left| \int_0^T \int_{\mathbb{R}^3} AB u \bar{v} dx dt \right| \lesssim_A T^{\delta} \|B\|_{U_W^2 L^2} \|A\|_{U_W^2 H^1} \|u\|_{\tilde{U}_A^2 H^2} \|v\|_{V_A^2 L^2} \tag{47}$$

To prove (46) we use a triple Littlewood-Paley decomposition to write

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^3} B \nabla u \bar{v} dx dt \right| \lesssim & \sum_{\mu \lesssim \lambda} I^T_{\lambda,\lambda,\mu}(\nabla u, v, B) + \sum_{\mu \ll \lambda} I^T_{\mu,\lambda,\lambda}(\nabla u, v, B) \\ & + \sum_{\mu \ll \lambda} I^T_{\lambda,\mu,\lambda}(\nabla u, v, B) \end{aligned}$$

Then for each term we use the corresponding bounds (39), and (40) with a $\ln \lambda$ correction coming from the use of Proposition 8. The summation with respect to μ and λ is straightforward.

For (47), by (13) the norms of A and B are l^2 summable with respect to unit spatial cubes. Hence without any restriction in generality we can assume that both A and B are supported in a unit cube Q. For A we use a Strichartz estimate, for B the energy and for u a pointwise bound. Finally, for v we interpolate the local energy estimates with the local Strichartz estimates to obtain

$$\|S_\lambda v\|_{L^2 L^{\frac{12}{5}}} \lesssim_A \lambda^{-\frac{1}{4}+\varepsilon} \|v\|_{V_A^2 L^2}$$

which leads to

$$\|v\|_{L^2 L^{\frac{12}{5}}} \lesssim_A \|v\|_{V_A^2 L^2}$$

Then we can estimate

$$\begin{aligned} \left| \int_0^T \int_Q AB u \bar{v} dx dt \right| & \lesssim T^{\frac{1}{4}} \|B\|_{L^\infty L^2} \|A\|_{L^4 L^{12}} \|u\|_{L^\infty} \|v\|_{L^2 L^{\frac{12}{5}}} \\ & \lesssim_A T^{\frac{1}{4}} \|B\|_{U_W^2 L^2} \|A\|_{U_W^2 H^1} \|u\|_{U_A^2 H^2} \|v\|_{V_A^2 L^2} \end{aligned}$$

□

We now return to the equation (44). By the definition of the $\tilde{U}_A^2 L^2$ norm and (45) we have

$$\|2(A_t \nabla - i A A_t)(1-\Delta_A)^{-1} v\|_{DU_A^2 L^2} \lesssim_A T^{\delta} \|v\|_{U_A^2 L^2} \tag{48}$$

By (9) it follows that we can solve (44) perturbatively in $U_A^2 L^2$ on short time intervals. This gives a solution $u = (1-\Delta_A)^{-1} v \in \tilde{U}_A^2 H^2$ for (19). Furthermore, we obtain the bound

$$\|v(t) - S(t,0)v(0)\|_{L^2} \lesssim_A T^{\delta} \|v(0)\|_{L^2}$$

where $S(t,s)$ is the evolution associated to (18). In particular this shows that $v \in C([0,1];L^2)$. An elliptic argument allows us to return to u and conclude that $u \in C([0,1];H^2)$. This concludes the proof of the H^2 well-posedness.

Finally, we show that $U^2_A H^2 = \tilde{U}^2_A H^2$ and the other two similar identities. Via the operator $I - \Delta_A$ these two spaces can be identified with the $U^2 L^2$ spaces associated to the equations (18), respectively (44). But by Lemma 3, these are equivalent due to (48).

□

Next we consider the well-posedness in H^{-2}, which is essentially dual to the H^2 well-posedness.

Proposition 18. *Let $A \in U^2_W H^1$ with $\nabla \cdot A = 0$. Then the equation (19) is well-posed in H^{-2}. In addition we have*

$$U^2_A H^{-2} = \tilde{U}^2_A H^{-2}, \qquad V^2_A H^{-2} = \tilde{V}^2_A H^{-2}, \qquad DU^2_A H^{-2} = \tilde{U}^2_A H^{-2}$$

with equivalent norms.

Proof. By Lemma 11 we can write the initial data u_0 as

$$u_0 = (1 - \Delta_A) v_0, \qquad v_0 \in L^2$$

Then we seek the solution u for (18) of the form $u = (1 - \Delta_A)v$. The equation for v is

$$(i\partial_t - \Delta_A)v = 2(1 - \Delta_A)^{-1}(A_t \nabla - iAA_t)v \tag{49}$$

To solve it we need the following counterpart to (45):

Lemma 19. *The operator $A_t \nabla - iAA_t$ satisfies the space-time bound*

$$\|(A_t \nabla - 2iAA_t)u\|_{DU^2_A H^{-2}} \lesssim_A T^\delta \|u\|_{\tilde{U}^2_A L^2} \tag{50}$$

Proof. By duality, (50) follows from the bounds

$$\left| \int_0^T \int_{\mathbb{R}^3} B \nabla u \bar{v} dx dt \right| \lesssim_A T^\delta \|u\|_{U^2_A L^2} \|v\|_{V^2_A H^2} \|B\|_{U^2_W L^2} \tag{51}$$

respectively

$$\left| \int_0^T \int_{\mathbb{R}^3} AB u \bar{v} dx dt \right| \lesssim_A T^\delta \|B\|_{U^2_W L^2} \|A\|_{U^2_W H^1} \|u\|_{U^2_A L^2} \|v\|_{V^2_A H^2} \tag{52}$$

These are almost identical to (46) and (47), and their proofs are essentially the same. □

The bound (50) allows us to solve the equation (49) perturbatively in $U^2_A L^2$ and obtain a solution $v \in C([0,1];L^2)$. This implies the H^{-2} solvability for (18). The second part of the proposition follows again from Lemma 3. □

Having the well-posedness result in H^2 and H^{-2} allows us to prove well-posedness in a range of intermediate spaces. Given a positive sequence $\{m(\lambda)\}_{\lambda=2^j}$ satisfying

$$0 < c < \frac{m(2\lambda)}{m(\lambda)} < C \tag{53}$$

we define the Sobolev type space $H(m)$ with norm

$$\|u\|_{H(m)}^2 = \sum_\lambda m^2(\lambda)\|S_\lambda u\|_{L^2}^2$$

The standard Sobolev spaces H^α are obtained by taking $m(\lambda) = \lambda^\alpha$.

We consider the solvability for (18) in $H(m)$ under a stronger condition for m, namely

$$\frac{1}{4} \leq \frac{m(2\lambda)}{m(\lambda)} \leq 4 \tag{54}$$

This guarantees that $H(m)$ is an intermediate space between H^{-2} and H^2.

We need to describe $H(m)$ in terms of H^{-2} and H^2. To measure functions which are localized at some frequency λ we can use the norm

$$\|u\|_{H_\lambda}^2 = \lambda^{-4}\|u\|_{H^2}^2 + \lambda^4\|u\|_{H^{-2}}^2$$

Ideally we would like to represent $H(m)$ as an almost orthogonal superposition of the H_λ spaces with the weights $m(\lambda)$. However, this does not work so well if $H(m)$ is "close" to either H^{-2} or H^2. Instead we need to select a subset of dyadic frequencies which achieves the desired result. We denote

$$m_\infty = \lim_{\lambda\to\infty} \lambda^2 m(\lambda)$$

On $2^{\mathbb{N}} \cup \{\infty\}$ we introduce the relation "$\prec$" by

$$\lambda \prec \mu \Leftrightarrow 2m(\mu) \geq m(\lambda)(\lambda^2\mu^{-2} + \mu^2\lambda^{-2}), \qquad \mu, \lambda < \infty$$

respectively

$$\infty \prec \mu \Leftrightarrow 2m(\mu) \geq \mu^{-2} m_\infty, \qquad \mu < \infty$$

Definition 20. *We say that a subset $\Lambda(m) \subset 2^{\mathbb{N}} \cup \{\infty\}$ is m-representative if (i) for each $\mu \in 2^{\mathbb{N}}$ there exists $\lambda \in \Lambda(m)$ so that $\lambda \prec \mu$ and (ii) for each $\mu \in \Lambda(m)$ there is at most one $\lambda \in \Lambda(m) \setminus \{\mu\}$ such that $\lambda \prec \mu$.*

Lemma 21. *If m satisfies* (54) *then an m-representative set $\Lambda(m)$ exists. In addition, for each $\mu \in 2^{\mathbb{N}}$ and $K \in \mathbb{N}$ we have*

$$|\{\lambda \in \Lambda(m); 2^K m(\mu) \geq m(\lambda)(\lambda^2\mu^{-2} + \mu^2\lambda^{-2})\}| \leq 4(K+4) \tag{55}$$

Proof. For each $\lambda \in 2^{\mathbb{N}} \cup \{\infty\}$ we denote

$$I_\lambda = \{\mu \in 2^{\mathbb{N}} \cup \{\infty\},\ \lambda \prec \mu\}$$

Due to (54) it is easy to see that I_λ is an interval,

$$I_\lambda = [\lambda^-, \lambda^+], \qquad \lambda^- \leq \lambda \leq \lambda^+$$

and the endpoints λ^- and λ^+ are nondecreasing functions of λ.

We construct the set $\Lambda(m)$ as an increasing sequence $\{\lambda_j\}$ in an iterative manner. λ_0 is chosen maximal so that $\lambda_0 \prec 1$. Iteratively, λ_{j+1} is chosen maximal so that $\lambda_{j+1} \prec 2\lambda_j^+$. Either this process continues for an infinite number of steps, or it stops at some step k with $\lambda_k = \infty$. The former occurs if $m_\infty = \infty$ and the latter if $m_\infty < \infty$.

The property (i) in Definition 20 is satisfied by construction. For (ii) we observe that $\lambda_{j+1} \geq 2\lambda_j^+$ therefore $\lambda_j \not\prec \lambda_{j+1}$. On the other hand by construction we have $\lambda_{j+2} \not\prec \lambda_j$.

For (55) suppose $\mu \leq \lambda_j$. Then

$$m(\mu) \geq \frac{1}{4}m(\lambda_j)\mu^2\lambda_j^{-2} \geq \frac{1}{8}m(\lambda_{j+2})\mu^2\lambda_{j+2}^{-2} \geq \cdots \geq 2^{-K-2}m(\lambda_{j+2K})\mu^2\lambda_{j+2K}^{-2}$$

A similar bound holds if we descend from μ, and the conclusion follows. $\square$

Since we allow $\infty \in \Lambda(m)$ we need the equivalent of H_λ in that case, which is defined by $H_\infty = H^{-2}$. We note that at the other extreme we have $H_1 = H^2$.

Lemma 22. *Let m satisfy (54), and $\Lambda(m)$ be an m-representative subset of $2^N \cup \{\infty\}$. Then*

$$\|u\|_{H(m)}^2 \approx \inf\{\sum_{\lambda\in\Lambda(m)} m(\lambda)^2\|u_\lambda\|_{H_\lambda}^2,\ u = \sum_{\lambda\in\Lambda(m)} u_\lambda\}$$

Proof. By Definition 20 we have a finite covering of $2^{\mathbb{N}}$ with intervals

$$2^{\mathbb{N}} \subset \bigcup_{\lambda\in\Lambda(m)} I_\lambda$$

We consider an associated partition of unity in the Fourier space,

$$1 = \sum_{\lambda\in\Lambda(m)} \chi_\lambda(\xi)$$

For $\mu \in I_\lambda$ we have $m(\mu) \approx m(\lambda)(\mu^2\lambda^{-2} + \lambda^2\mu^{-2})$ therefore we obtain

$$\|\chi_\lambda(D_x)u\|_{H(m)} \approx m(\lambda)\|\chi_\lambda(D_x)u\|_{H_\lambda}$$

and the "$\gtrsim$" inequality follows. For the reverse we use (55), which shows that the series $\sum u_\lambda$ is almost orthogonal in $H(m)$,

$$\langle u_{\lambda_i}, u_{\lambda_j}\rangle_{H(m)} \lesssim 2^{-|i-j|}m(\lambda_i)m(\lambda_j)\|u_{\lambda_i}\|_{H_{\lambda_i}}\|u_{\lambda_j}\|_{H_{\lambda_j}}$$

$\square$

Finally we consider the well-posedness of (18) in $H(m)$.

Proposition 23. *a) Assume that the sequence m satisfies (54). Then the equation (18) is well-posed in $H(m)$.*

b) Furthermore, for each $u \in U_A^2H(m)$ there is a representation

$$u = \sum_{\lambda\in\Lambda(m)} u_\lambda$$

with

$$\sum_{\lambda \in \Lambda(m)} m^2(\lambda) \left(\lambda^{-4} \|u_\lambda\|^2_{U^2_A H^2} + \lambda^4 \|u_\lambda\|^2_{U^2_A H^{-2}} \right) \lesssim_A \|u\|^2_{U^2_A H(m)} \tag{56}$$

c) The following duality relation holds:

$$(DU^2_A H(m))^* = V^2_A H(m^{-1})$$

Proof. a) We consider a dyadic decomposition of the initial data

$$u_0 = \sum_{\lambda \in \Lambda(m)} \chi_\lambda(D_x) u_0$$

and denote by u_λ the solutions to (18) with initial data $S_\lambda u_0$. Then

$$u = \sum_{\lambda \in \Lambda(m)} u_\lambda$$

We can measure u_λ in both H^2 and H^{-2},

$$\lambda^{-2} \|u_\lambda\|_{C([0,1],H^2)} + \lambda^2 \|u_\lambda\|_{C([0,1],H^{-2})} \lesssim_A \|S_\lambda u_0\|_{L^2}$$

After summation this gives

$$\sum_{\lambda \in \Lambda(m)} m^2(\lambda) \left(\lambda^{-4} \|u_\lambda\|_{C([0,1],H^2)} + \lambda^4 \|u_\lambda\|_{C([0,1],H^{-2})} \right) \lesssim_A \|u_0\|^2_{H(m)}$$

which, by (54), implies that $u \in C([0,1], H(m))$ and

$$\|u\|_{C([0,1],H(m))} \lesssim_A \|u_0\|_{H(m)}$$

b) It suffices to consider the case when u is an $U^2_A H(m)$ atom. Then we consider a decomposition as in part (a) for each of the steps, and the conclusion follows.

c) This is a direct consequence of (8). □

Finally, we can interpolate the properties from $U^2_A H^2$ and $U^2_A H^{-2}$ to obtain properties for $U^2 H(m)$. Indeed, we have the following

Proposition 24. *Let $A \in U^2_W H^1$ with $\nabla \cdot A = 0$. Assume that m, m_1, m_2 satisfy* (54). *Then the following estimates hold:*

(i) For p, q as in (20) *we have the Strichartz estimate*

$$\|S_\lambda u\|_{L^p(0,T;L^q)} \lesssim_A T^{\frac{\delta}{p}} m(\lambda)^{-1} \lambda^{\frac{1}{p}} \|u\|_{U^2_A H(m)} \tag{57}$$

(ii) Local energy estimates. For any spatial cube Q of size 1 *we have:*

$$\|S_\lambda u\|_{L^2([0,T] \times Q)} \lesssim_A T^\delta m(\lambda)^{-1} \lambda^{-\frac{1}{2}+\epsilon} \|u_0\|_{\tilde{U}^2_A H(m)} \tag{58}$$

(iii) Local Strichartz estimates. For any spatial cube Q of size 1 *we have:*

$$\|S_\lambda u\|_{L^2(0,1;L^6(B))} \lesssim_A T^\delta m(\lambda)^{-1} \lambda^\epsilon \|u_0\|_{U^2_A H(m)} \tag{59}$$

(iv) Trilinear estimates. For $\mu \le \lambda$ we have

$$|I^T_{\lambda,\lambda,\mu}(u,v,B)| \lesssim_A \frac{T^\delta \lambda^\epsilon \min(1, \mu\lambda^{-\frac{1}{2}})}{m_1(\lambda)m_2(\lambda)} \|u\|_{U^2_A H(m_1)} \|v\|_{U^2_A H(m_2)} \|B\|_{U^2_W L^2} \tag{60}$$

while if $\mu \ll \lambda$ then

$$|I^T_{\lambda,\mu,\lambda}(u,v,B)| \lesssim_A \frac{T^\delta \lambda^{-\frac{1}{2}+\epsilon}\mu^{\frac{1}{2}}}{m_1(\lambda)m_2(\mu)} \|u\|_{U^2_A H(m_1)} \|v\|_{U^2_A H(m_2)} \|B\|_{U^2_W L^2} \tag{61}$$

Due to the representation in Proposition 23(b) this result is a straightforward consequence of the similar results in H^2 and H^{-2}.

5. Proof of Theorem 1

We first establish an a-priori estimate for regular (H^2) solutions of (3). For this we need to consider the two nonlinear expressions on the right hand side of (3). We begin with the Schrödinger nonlinearity:

Lemma 25. *For $\beta > \frac{1}{2}$ and m satisfying* (54) *we have*

$$\|\phi v\|_{DU^2_A(0,T;H(m))} \lesssim_A T^\delta \|v\|_{U^2_A H(m)} \|u\|^2_{U^2_A H^\beta} \tag{62}$$

Proof. By duality the above bound is equivalent to the quadrilinear estimate

$$\left| \int_0^T\!\!\int_{\mathbb{R}^3} \Delta^{-1}(u_1\bar{u}_2)u_3\bar{u}_4 dxdt \right| \lesssim_A T^\delta \|u_1\|_{U^2_A H^\beta} \|u_2\|_{U^2_A H^\beta} \|u_3\|_{U^2_A H(m)} \|u_4\|_{V^2_A H(\frac{1}{m})}$$

After a simultaneous Littlewood-Paley decomposition of the three factors $\Delta^{-1}(u_1\bar{u}_2)$, u_3 and u_4 we need to consider the following three sums:

$$S_{lhh} = \sum_{\mu \le \lambda} \int_0^T\!\!\int_{\mathbb{R}^3} \Delta^{-1} S_\mu(u_1\bar{u}_2) S_\lambda u_3 S_\lambda \bar{u}_4 dxdt$$

$$S_{hlh} = \sum_{\mu \ll \lambda} \int_0^T\!\!\int_{\mathbb{R}^3} \Delta^{-1} S_\lambda(u_1\bar{u}_2) S_\mu u_3 S_\lambda \bar{u}_4 dxdt$$

$$S_{hhl} = \sum_{\mu \ll \lambda} \int_0^T\!\!\int_{\mathbb{R}^3} \Delta^{-1} S_\lambda(u_1\bar{u}_2) S_\lambda u_3 S_\mu \bar{u}_4 dxdt$$

The first sum can be estimated using Strichartz and energy estimates as follows:

$$\begin{aligned}
|S_{lhh}| &\lesssim \sum_\mu \mu^{-2} \|S_\mu(u_1\bar{u}_2)\|_{L^1L^\infty} \Big\| \sum_{\lambda \ge \mu} S_\lambda u_3 S_\lambda \bar{u}_4 \Big\|_{L^\infty L^1} \\
&\lesssim \sum_\mu \mu^{-1} \|S_\mu(u_1\bar{u}_2)\|_{L^1L^3} \sup_{t\in[0,T]} \sum_{\lambda\ge\mu} \|S_\lambda u_3(t)\|_{L^2} \|S_\lambda u_4(t)\|_{L^2} \\
&\lesssim \|u_1\|_{L^2L^6} \|u_2\|_{L^2L^6} \sup_{t\in[0,T]} \|u_3(t)\|_{H(m)} \|u_4(t)\|_{H(m^{-1})} \\
&\lesssim_A T^\delta \|u_1\|_{U^2_A H^\beta} \|u_2\|_{U^2_A H^\beta} \|u_3\|_{U^2_A H(m)} \|u_4\|_{V^2_A H(m^{-1})}
\end{aligned}$$

The second and third sums are similar. Using the Strichartz estimates we obtain

$$
\begin{aligned}
|S_{hlh}| &\lesssim \sum_{\mu \ll \lambda} \lambda^{-2} \|S_\lambda(u_1\bar{u}_2)\|_{L^2L^2} \|S_\mu u_3\|_{L^2L^\infty} \|S_\lambda \bar{u}_4\|_{L^\infty L^2} \\
&\lesssim_A T^\delta \sum_{\mu \ll \lambda} \frac{\mu m(\lambda)}{\lambda^{\frac{3}{2}} m(\mu)} \|S_\lambda(u_1\bar{u}_2)\|_{L^2 L^{\frac{3}{2}}} \|u_3\|_{U^2_A H(m)} \|u_4\|_{V^2_A H(m^{-1})}
\end{aligned}
$$

At least one of u_1 or u_2, say u_1, must have frequency at least λ. Then we continue with

$$
\begin{aligned}
|S_{hlh}| &\lesssim_A T^\delta \sum_{\mu \ll \lambda} \frac{\mu m(\lambda)}{\lambda^{\frac{3}{2}} m(\mu)} \|u_1\|_{L^\infty L^2} \|\bar{u}_2\|_{L^2L^6} \|u_3\|_{U^2_A H(m)} \|u_4\|_{V^2_A H(m^{-1})} \\
&\lesssim_A T^\delta \sum_{\mu \ll \lambda} \frac{\mu^2 m(\lambda)}{\lambda^2 m(\mu)} \frac{\lambda^{\frac{1}{2}-\beta}}{\mu} \|u_1\|_{U^2_A H^\beta} \|\bar{u}_2\|_{U^2_A H^\beta} \|u_3\|_{U^2_A H(m)} \|u_4\|_{V^2_A H(m^{-1})}
\end{aligned}
$$

By (54) the first fraction is less than one therefore the summation with respect to λ and μ is straightforward. $\square$

Next we consider the wave nonlinearity. If m is as in (53) then the linear wave equation is well-posed in $H(m)$, and can easily define the corresponding spaces $U^2_W H(m)$, $V^2_W H(m)$ respectively $DU^2_W H(\lambda^{-1} m)$. The next result asserts that in effect the contribution of the wave nonlinearity is one half of a derivative better than the solution to the Schrödinger equation. Here we impose an additional condition on m, namely

$$
\frac{m(\lambda)}{m(\mu)} \geq \frac{\lambda^\beta}{\mu^\beta}, \qquad \lambda > \mu \tag{63}
$$

which guarantees that the $H(m)$ norm is at least as strong as the H^β norm.

Lemma 26. *a) Let $\beta > \frac{1}{2}$ and m satisfying* (54) *and* (63)*. Then*

$$
\begin{aligned}
\|P(\bar{u}\nabla_A u)\|_{DU^2_W(0,T;H(\lambda^{-\frac{1}{2}} m))} \lesssim_A \; & T^\delta (\|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \\
& + \|u\|^2_{U^2_A H^\beta} \|A\|_{U^2_W H(m)})
\end{aligned} \tag{64}
$$

b) For $\beta > \frac{3}{4}$ we have

$$
\|P(\bar{u}\nabla_A v)\|_{DU^2_W(0,T;H^{-\frac{1}{2}})} \lesssim_A T^\delta \|u\|_{U^2_A L^2} \|v\|_{U^2_A H^\beta} \tag{65}
$$

Proof. a) By duality we have two estimates to prove. The first is

$$
\left| \int_0^T \!\! \int_{\mathbb{R}^3} \bar{u} \nabla u B dx dt \right| \lesssim_A T^\delta \|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \|B\|_{V^2_W H(\lambda^{\frac{1}{2}}/m)} \tag{66}
$$

with divergence free B. The second is

$$
\begin{aligned}
\left| \int_0^T \!\! \int_{\mathbb{R}^3} \bar{u} u A B dx dt \right| \lesssim_A \; & T^\delta \|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \|A\|_{V^2_W H^1} \|B\|_{V^2_W H(\lambda^{\frac{1}{2}}/m)} \\
& + T^\delta \|u\|_{U^2_A H^\beta} \|u\|_{U^2_A H^\beta} \|A\|_{V^2_W H(m)} \|B\|_{V^2_W H(\lambda^{\frac{1}{2}}/m)}
\end{aligned} \tag{67}
$$

Consider (66). Since B is divergence free, it follows that the gradient can be placed either on u or on $\bar{u}$. Hence using a simultaneous trilinear Littlewood-Paley decomposition of the three factors we reduce the problem to estimating the following two terms:

$$S_{hhl} = \sum_{\lambda} \sum_{\mu \lesssim \lambda} |I^T_{\lambda,\lambda,\mu}(u, \nabla u, B)|, \qquad S_{hlh} = \sum_{\lambda} \sum_{\mu \ll \lambda} |I^T_{\lambda,\mu,\lambda}(u, \nabla u, B)|$$

We use (60) and Lemma 8 to estimate the first term:

$$S_{hhl} \lesssim_A T^\delta \sum_{\lambda} \sum_{\mu \lesssim \lambda} \frac{\lambda^\epsilon m(\mu)}{\lambda^\beta \mu^{\frac{1}{2}} m(\lambda)} \|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \|B\|_{V^2_W H(\lambda^{\frac{1}{2}}/m)}$$

The bound (63) insures that the summation is straightforward if ε is chosen sufficiently small. For the second term we use (61) and Lemma 8:

$$S_{hlh} \lesssim_A T^\delta \sum_{\lambda} \sum_{\mu \lesssim \lambda} \frac{\mu^{\frac{3}{2}-\beta}}{\lambda^{1-\varepsilon}} \|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \|B\|_{V^2_W H(\lambda^{\frac{1}{2}}/m)}$$

which is again summable if ε is sufficiently small.

Next we turn our attention to (67). Using a Littlewood-Paley decomposition for all terms it suffices to consider factors of type

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = \left| \int_0^T \int_{\mathbb{R}^3} S_{\lambda_1} \bar{u}\, S_{\lambda_2} u\, S_{\lambda_3} A\, S_{\lambda_4} B dx dt \right|$$

and prove that for some $\delta > 0$ they satisfy the bound

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \lesssim_A \lambda_1^{-\delta} \lambda_2^{-\delta} \lambda_3^{-\delta} \lambda_4^{-\delta} \cdot RHS((67)) \tag{68}$$

We begin with a weaker bound which follows directly from Strichartz estimates, namely

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \lesssim_A T^{\frac{1}{3}} \lambda_1^{-\frac{1}{4}} \lambda_2^{-\frac{1}{4}} \lambda_3^{-\frac{2}{3}} \lambda_4^N \|u\|^2_{U^2_A H^\beta} \|A\|_{U^2_W H^1} \|B\|_{L^\infty L^2}$$

Arguing as in Lemma 8, this allows us to replace (68) with

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \lesssim_A \lambda_1^{-\delta} \lambda_2^{-\delta} \lambda_3^{-\delta} \lambda_4^{-\delta} \cdot \text{ modified } RHS((67)) \tag{69}$$

where we have replaced $V^2_W H(\lambda^{\frac{1}{2}}/m)$ space with the similar U^2 space in the right hand side of (67).

Due to (13) both wave factors are l^2 summable with respect to unit spatial cubes therefore it is enough to estimate the above integral on a unit cube Q. Also we must have $\lambda_4 \leq \max\{\lambda_1, \lambda_2, \lambda_3\}$. Hence we consider two cases.

If $\max\{\lambda_1, \lambda_2, \lambda_3\} = \lambda_1$ (or λ_2) then we estimate

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \leq T^{\frac{1}{6}} \|S_{\lambda_1} u\|_{L^4 L^3} \|S_{\lambda_2} u\|_{L^2 L^\infty} \|S_{\lambda_3} A\|_{L^3 L^6} \|S_{\lambda_4} B\|_{L^\infty L^2}$$

$$\leq T^{\frac{1}{6}} \lambda_1^{-\frac{1}{2}+\varepsilon} \lambda_2^{\frac{1}{2}+\varepsilon-\beta} \lambda_3^{-\frac{1}{6}} \frac{\lambda_1^{\frac{1}{2}} m(\lambda_4)}{\lambda_4^{\frac{1}{2}} m(\lambda_1)} \|u\|_{U^2_A H(m)} \|u\|_{U^2_A H^\beta} \|A\|_{V^2_W H^1} \|B\|_{U^2_W H(\frac{\sqrt{\lambda}}{m})}$$

By (63) the fraction above is less than one therefore for small enough ε the bound (69) follows.

The second case is when $\max\{\lambda_1,\lambda_2,\lambda_3\}=\lambda_3$. Then we estimate

$$I^T_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \leq T^{\frac{1}{12}}\|S_{\lambda_1}u\|_{L^4L^6}\|S_{\lambda_2}u\|_{L^2L^\infty}\|S_{\lambda_3}A\|_{L^6L^3}\|S_{\lambda_4}B\|_{L^\infty L^2}$$

$$\lesssim_A T^{\frac{1}{12}}\lambda_1^{\frac{1}{3}+\varepsilon-\beta}\lambda_2^{\frac{1}{2}+\varepsilon-\beta}\lambda_3^{-\frac{1}{6}}\frac{\lambda_3^{\frac{1}{2}}m(\lambda_4)}{\lambda_4^{\frac{1}{2}}m(\lambda_3)}\|u\|_{U^2_AH^\beta}\|u\|_{U^2_AH^\beta}\|A\|_{U^2_WH(m)}\|B\|_{U^2_WH(\frac{\sqrt{\lambda}}{m})}$$

and (69) again follows.

b) By duality we have two estimates to prove. The first one is trilinear,

$$\left|\int_0^T\int_{\mathbb{R}^3}\bar{u}\nabla vBdxdt\right| \lesssim_A T^\delta\|u\|_{U^2_AL^2}\|v\|_{U^2_AH^\beta}\|B\|_{V^2_WH^{\frac{1}{2}}} \tag{70}$$

with a divergence free B. The second one is quadrilinear,

$$\left|\int_0^T\int_{\mathbb{R}^3}\bar{u}uABdxdt\right| \lesssim_A T^\delta\|u\|_{U^2_AH^\beta}\|v\|_{U^2_AL^2}\|A\|_{U^2_WH^1}\|B\|_{V^2_WH^{\frac{1}{2}}} \tag{71}$$

Since B is divergence free, in (70) we can place the gradient on either u or v. The argument is similar to the one in part (a), using (60) and (61), as well as Lemma 8 in order to substitute the V^2 norm by the U^2 norm. The new restriction $\beta>\frac{3}{4}$ arises in the case when the low frequency is on B. Indeed, if $\mu\ll\lambda$ then by (60) we have

$$|I^T_{\lambda,\lambda,\mu}(u,\nabla v,B)|\lesssim_A T^\delta\lambda^{1-\beta+\epsilon}\mu^{-\frac{1}{2}}\min\{1,\mu\lambda^{-\frac{1}{2}}\}\|u\|_{U^2_AL^2}\|v\|_{U^2_AH^\beta}\|B\|_{V^2_WH^{\frac{1}{2}}}$$

The worst case is when $\mu=\lambda^{\frac{1}{2}}$, when the coefficient above is $T^\delta\lambda^{\frac{3}{4}-\beta+\varepsilon}$.

The estimate (71) is also proved as in part (a). Indeed, by Corollary 7 we can substitute the V^2 space by U^2 at the expense of losing ε derivatives. Because of the finite speed of propagation for the wave equation, see (13), we can reduce the problem to the case when A,B are supported in a unit cube Q. There we can use the Strichartz estimates for the wave equation, respectively the local Strichartz estimates for the Schrödinger equation.

□

The next step in the proof of the theorem is to establish an a-priori H^1 estimate for H^2 solutions. This is obtained in terms of the conserved quantities in our problem, namely E and Q.

Proposition 27. *Let (u,A) be an H^2 solution for* (3) *in some time interval $[0,T_0]$ with $T_0\leq 1$. Then*

$$\|u\|_{U^2_A(0,T_0;H^1)}+\|A\|_{U^2_W(0,T_0;H^1)}\leq c(E,Q)$$

Proof. We use a bootstrap argument. Since $u,A\in L^\infty H^2$, we can easily estimate the wave nonlinearity and obtain $\Box A\in L^\infty H^1$. This implies that

$A \in U^2_W H^1$, and, in addition, that the function

$$T \to \|A\|_{U^2_W(0,T;H^1)}$$

is continuous and satisfies

$$\lim_{T\to 0} \|A\|_{U^2_W(0,T;H^1)} = \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2}$$

A similar argument applies in the case of the Schrödinger equation.

By (9) we estimate in the Schrödinger equation

$$\|u\|_{U^2_A(0,T;H^1)} \lesssim \|u(0)\|_{H^1} + \|iu_t - \Delta_A u\|_{DU^2_A(0,T;H^1)}.$$

Then by (62) we obtain

$$\|u\|_{U^2_A(0,T;H^1)} \leq C^1_A(\|u(0)\|_{H^1} + T^\delta \|u\|^3_{U^2_A(0,T;H^1)})$$

Similarly we can use (64) to obtain a bound for the wave equation

$$\|A\|_{U^2_W(0,T;H^1)} \leq \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2} + T^\delta C^2_A \|u\|^2_{U^2_A(0,T;H^1)}$$

We multiply the first equation by $c^1_A = (C^1_A)^{-1}$ and add to the second equation to obtain

$$\begin{aligned}\|A\|_{U^2_W(0,T;H^1)} + c^1_A\|u\|_{U^2_A(0,T;H^1)} \leq \|u(0)\|_{H^1} + \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2}\\ + T^\delta C^3_A(1 + \|u\|^3_{U^2_A(0,T;H^1)})\end{aligned}$$

We make the bootstrap assumption

$$c^1_A\|u\|_{U^2_A(0,T;H^1)} + \|A\|_{U^2_W(0,T;H^1)} \leq 2 + \|u(0)\|_{H^1} + \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2}$$

Then the previous bound implies that

$$\begin{aligned}c^1_A\|u\|_{U^2_A(0,T;H^1)} + \|A\|_{U^2_W(0,T;H^1)} \leq \|u(0)\|_{H^1} + \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2}\\ + T^\delta C(E,Q)(1 + \|u\|^3_{U^2_A(0,T;H^1)})\end{aligned}$$

This shows that for $T \leq T_0(E,Q)$ we have

$$\|u\|_{U^2_A(0,T;H^1)} + \|A\|_{U^2_W(0,T;H^1)} \leq 1 + \|u(0)\|_{H^1} + \|A(0)\|_{H^1} + \|A_t(0)\|_{L^2} \tag{72}$$

improving our bootstrap assumption.

Hence a continuity argument shows that (72) holds without any bootstrap assumption. The conclusion of the proposition follows by summing up with respect to $T_0(E,Q)$ time intervals.

□

The next step is to establish an a-priori H^2 bound with constants which depend only on the H^1 size of the data.

Proposition 28. *Let (u, A) be an H^2 solution for (3) in some time interval $[0, T_0]$ with $T_0 \leq 1$. Then*

$$\|u\|_{U^2_A H^2} + \|A\|_{U^2_W H^2} \leq c(E,Q)(\|u_0\|_{H^2} + \|A_0\|_{H^2} + \|A_1\|_{H^1})$$

Proof. The argument is similar to the one above. □

Given the local well-posedness result in H^2 proved in earlier work [9], we can iterate the argument and conclude that the H^2 solutions are global.

Finally, our last apriori estimate is in intermediate spaces:

Proposition 29. *Let m be a weight which satisfies* (54) *and* (63). *Let (u, A) be an H^2 solution for* (3) *in the time interval* $[0, 1]$. *Then*

$$\|u\|_{U_A^2 H(m)} + \|A\|_{U_W^2 H(m)} \leq c(E, Q)(\|u_0\|_{H(m)} + \|A_0\|_{H(m)} + \|A_1\|_{H(\lambda^{-1}m)})$$

Proof. The argument is again similar to the one above. □

In order to obtain H^1 solutions and to study the dependence of the solutions on the initial data we need to obtain estimates for differences of solutions. Given a solution (u, A) to (3) we consider the corresponding linearized problem

$$\begin{cases} iv_t - \Delta_A v = 2iB\nabla u + 2ABu + \phi v + \Delta^{-1}(\Re u\bar{v})u \\ \\ \Box B = P(\bar{v}\nabla_A u + \bar{u}\nabla_A v + B|u|^2) \end{cases} \tag{73}$$

Our main estimate for the linearized problem is

Proposition 30. *Let (u, A) be an H^2 solution for* (3) *in the time interval* $[0, 1]$. *Then the linearized problem* (73) *is well-posed in $L^2 \times H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$ uniformly with respect to (u, A) in a bounded set in the energy space,*

$$\|v\|_{U_A^2 L^2} + \|B\|_{U_W^2 H^{\frac{1}{2}}} \leq c(E, Q)(\|v_0\|_{L^2} + \|B_0\|_{H^{\frac{1}{2}}} + \|B_1\|_{H^{-\frac{1}{2}}}) \tag{74}$$

Proof. The conclusion follows iteratively in short time intervals provided that we obtain appropriate estimates for the terms on the right:

$$\|2iB\nabla u+2ABu+\phi v+\Delta^{-1}(\Re u\bar{v})u\|_{DU_A^2 L^2} \lesssim T^\delta c(E, Q)(\|v\|_{U_A^2 L^2}+\|B\|_{U_W^2 H^{\frac{1}{2}}})$$

respectively

$$\|P(\bar{v}\nabla_A u + \bar{u}\nabla_A v + iB|u|^2)\|_{DU_W^2 H^{-\frac{1}{2}}} \lesssim T^\delta c(E, Q)(\|v\|_{U_A^2 L^2} + \|B\|_{U_W^2 H^{\frac{1}{2}}})$$

These in turn follow by duality from the trilinear and quadrilinear bounds

$$\left|\int_0^T\int_{\mathbb{R}^3} B\nabla u_1 \bar{u}_2 dxdt\right| \lesssim_A T^\delta \|B\|_{U_W^2 H^{\frac{1}{2}}} \|u_1\|_{U_A^2 H^1} \|u_2\|_{V_A^2 L^2} \tag{75}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} ABu_1 \bar{u}_2 dxdt\right| \lesssim_A T^\delta \|A\|_{U_W^2 H^1} \|B\|_{U_W^2 H^{\frac{1}{2}}} \|u_1\|_{U_A^2 H^1} \|u_2\|_{V_A^2 L^2} \tag{76}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} \Delta^{-1}(u_1\bar{u}_2)u_3\bar{u}_4 dxdt\right| \lesssim_A T^\delta \|u_1\|_{U_A^2 H^1} \|u_2\|_{U_A^2 H^1} \|u_3\|_{U_A^2 L^2} \|u_4\|_{V_A^2 L^2} \tag{77}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} \Delta^{-1}(u_1\bar{u}_2)u_3\bar{u}_4 dxdt\right| \lesssim_A T^\delta \|u_1\|_{U_A^2 H^1} \|u_2\|_{U_A^2 L^2} \|u_3\|_{U_A^2 H^1} \|u_4\|_{V_A^2 L^2} \tag{78}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} B\nabla u_1\bar{u}_2dxdt\right| \lesssim_A T^\delta\|B\|_{V^2_W H^{\frac12}}\|u_1\|_{U^2_A H^1}\|u_2\|_{U^2_A L^2} \tag{79}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} B\nabla u_1\bar{u}_2dxdt\right| \lesssim_A T^\delta\|B\|_{V^2_W H^{\frac12}}\|u_1\|_{U^2_A L^2}\|u_2\|_{U^2_A H^1} \tag{80}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} ABu_1\bar{u}_2dxdt\right| \lesssim_A T^\delta\|A\|_{U^2_W H^1}\|B\|_{V^2_W H^{\frac12}}\|u_1\|_{U^2_A H^1}\|u_2\|_{U^2_A L^2} \tag{81}$$

$$\left|\int_0^T\int_{\mathbb{R}^3} ABu_1\bar{u}_2dxdt\right| \lesssim_A T^\delta\|A\|_{U^2_W H^{\frac12}}\|B\|_{V^2_W H^{\frac12}}\|u_1\|_{U^2_A H^1}\|u_2\|_{U^2_A H^1} \tag{82}$$

The quadrilinear mixed bounds (76), (81), (82) follow trivially from the Strichartz estimates. For (76) for instance we have

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^3} ABu_1\bar{u}_2dxdt\right| &\lesssim T^{\frac{1}{12}}\|A\|_{L^3L^8}\|B\|_{L^4}\|u_1\|_{L^3L^8}\|u_2\|_{L^\infty L^2}\\ &\lesssim_A T^{\frac{1}{12}}\|A\|_{U^2_W H^1}\|B\|_{U^2_W H^{\frac12}}\|u_1\|_{U^2_A H^1}\|u_2\|_{V^2_A L^2}\end{aligned}$$

We note that there is significant room for improvement in this computation by localizing first to the unit spatial scale and then using the local Strichartz estimates for the Schrödinger equation.

The quadrilinear Schrödinger bound (77) corresponds to the particular choice $m(\lambda) = 1$ and $\beta = 1 > \frac12$ in (62). For (78) we can write

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^3} \Delta^{-1}(u_1\bar{u}_2)u_3\bar{u}_4dxdt\right| &\lesssim \|u_1u_2\|_{L^2L^{\frac32}}\|u_3u_4\|_{L^2L^{\frac32}}\\ &\lesssim T^{\frac13}\|u_1\|_{L^3L^6}\|u_2\|_{L^\infty L^2}\|u_3\|_{L^3L^6}\|u_4\|_{L^\infty L^2}\\ &\lesssim_A T^{\frac13}\|u_1\|_{U^2_A H^1}\|u_2\|_{U^2_A L^2}\|u_3\|_{U^2_A H^1}\|u_4\|_{V^2_A L^2}\end{aligned}$$

Finally, the bounds (79) and (80) are identical since div $B = 0$ and correspond to (70). (75) is essentially the same estimate.

□

Proof of Theorem 1, conclusion:

By Proposition 30 we can obtain a weak Lipschitz dependence result for H^2 solutions (u_1, A_1) and (u_2, A_2) to (3),

$$\begin{aligned}&\|u_1 - u_2\|_{L^\infty L^2} + \|A_1 - A_2\|_{U^2_W H^{\frac12}} \leq c(E_1, Q_1, E_2, Q_2)\\ &(\|(u_1 - u_2)(0)\|_{L^2} + \|(A_1 - A_2)(0)\|_{H^{\frac12}} + \|(A_1 - A_2)_t(0)\|_{H^{-\frac12}})\end{aligned} \tag{83}$$

We use this in order to construct solutions to (3) for H^1 initial data. Given $(u_0, A_0, A_1) \in H^1 \times H^1 \times L^2$ we consider a sequence of H^2 initial data

$$(u_0^n, A_0^n, A_1^n) \to (u_0, A_0, A_1) \text{ in } H^1 \times H^1 \times L^2$$

The sequence (u_0^n, A_0^n, A_1^n) is compact in $H^1 \times H^1 \times L^2$, therefore we can bound them uniformly in a stronger norm,

$$\|(u_0^n, A_0^n, A_1^n)\|_{H(m)\times H(m)\times H(\lambda^{-1}m)} \leq M$$

where $m(\lambda) \geq \lambda$ satisfies (54) and (63) and in addition

$$\lim_{\lambda\to\infty} \lambda^{-1} m(\lambda) = \infty$$

By Proposition 29 we obtain a uniform bound

$$\|u^n\|_{L^\infty H(m)} + \|A^n\|_{U_W^2 H(m)} \leq M$$

On the other hand, (83) shows that the solutions (u^n, A^n) have a limit in a weaker topology,

$$(u^n, A^n) \to (u, A) \qquad \text{in } L^\infty L^2 \times U_W^2 H^{\frac{1}{2}}$$

Combining the two bounds above we obtain strong convergence in H^1,

$$(u^n, A^n) \to (u, A) \qquad \text{in } L^\infty H^1 \times U_W^2 H^1$$

In addition, u will also satisfy the same Strichartz estimates as u^n. Passing to the limit in the equation (3) we easily see that (u, A) is a solution. Due to the weak Lipschitz dependence it is also the unique uniform limit of strong solutions. Due to the Strichartz estimates we can bound the nonlinear term ϕu in the Schrödinger equation as in Lemma 62. Then it also follows that $u \in U_A^2 H^1$.

The weak Lipschitz dependence (83) carries over to H^1 solutions, as well as the bounds in Propositions 27,29. Then the same argument as above gives the continuous dependence on the initial data.

6. Wave packets for Schrödinger operators with rough symbols

An essential part of this article is devoted to understanding the properties of the (95) flow at frequency λ on λ^{-1} time intervals. As it turns out, for many estimates the parameter λ can be factored out by rescaling. This is why in this section we consider a more general equation of the form

$$iu_t - \Delta u + a^w(t, x, D)u = 0, \qquad u(0) = u_0 \tag{84}$$

which we study on a unit time scale. Here a is a real symbol which is roughly smooth on the unit scale.

For such a problem one seeks to obtain a wave packet parametrix, i.e. to write solutions as almost orthogonal superpositions of wave packets, where the wave packets are localized both in space and in frequency on the unit scale. The simplest setup is to assume uniform bounds on a of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \leq c_{\alpha\beta}, \qquad |\alpha| + |\beta| \geq k$$

An analysis of this type has been carried out in [10], [11], [7]. If $k = 2$ then one obtains a wave packet parametrix where the packets travel along the Hamilton flow. If $k = 1$ the geometry simplifies, and the Hamilton flow

stays close to the flow for $a = 0$; however, a still affects a time modulation factor arising in the solutions. Finally if $k = 0$ then the $a^w(t, x, D)$ term is purely perturbative.

For the operators arising in the present paper the above uniform bounds on a are too strong, and need to be replaced by integral bounds of the form

$$\int_0^1 |\partial_x^\alpha \partial_\xi^\beta a(t, x^t, \xi^t)| \leq c_{\alpha\beta}, \qquad |\alpha| + |\beta| \geq k$$

where $t \to (x^t, \xi^t)$ is the associated Hamilton flow. The case $k = 2$ has been considered in [8]; as proved there, the Hamilton flow is bilipschitz and a wave packet parametrix can be constructed. The case $k = 0$ was considered in [1]; then the term $a^w(t, x, D)$ is perturbative, and one may use the $a = 0$ Hamilton flow in the above condition.

In the present article we need to deal with the case $k = 1$. This corresponds to a Hamilton flow which is close to the $a = 0$ flow. However, the term $a^w(t, x, D)$ is nonperturbative, and contributes a time modulation factor along each packet. Given these considerations, we consider the following assumption on the symbol a:

$$\sup_{x,\xi} \int_0^1 |\partial_x^\alpha \partial_\xi^\beta a(t, x + 2t\xi, \xi)| dt \leq \epsilon c_{\alpha,\beta} \qquad |\alpha| + |\beta| \geq 1 \tag{85}$$

Let $(x_0, \xi_0) \in \mathbb{R}^{2n}$. To describe functions which are localized in the phase space on the unit scale near (x_0, ξ_0) we use the norm:

$$H^{N,N}_{x^0,\xi^0} := \{f : \langle D - \xi^0 \rangle^N f \in L^2,\ \langle x - x^0 \rangle^N f \in L^2\}$$

We work with the lattice $\mathbb{Z}^n$ both in the physical and Fourier space. We consider a partition of unity in the physical space,

$$\sum_{x_0 \in \mathbb{Z}^n} \phi_{x_0} = 1 \qquad \phi_{x_0}(x) = \phi(x - x_0)$$

where ϕ is a smooth bump function with compact support. We use a similar partition of unity on the Fourier side:

$$\sum_{\xi_0 \in \mathbb{Z}^n} \varphi_{\xi_0} = 1, \qquad \varphi_{\xi_0}(\xi) = \varphi(\xi - \xi_0)$$

An arbitrary function u admits an almost orthogonal decomposition

$$u = \sum_{(x_0,\xi_0) \in \mathbb{Z}^{2n}} u_{x_0,\xi_0}, \qquad u_{x_0,\xi_0} = \varphi_{\xi_0}(D)(\phi_{x_0} u)$$

so that

$$\sum_{(x_0,\xi_0) \in \mathbb{Z}^{2n}} \|u_{x_0,\xi_0}\|^2_{H^{N,N}_{x^0,\xi^0}} \lesssim \|u\|^2_{L^2} \tag{86}$$

We remark that a continuous analog of the above discrete decomposition can be obtained using the Bargman transform.

We first establish that the Hamilton flow is close to the Hamilton flow with $a = 0$:

Lemma 31. *Assume that* (85) *holds with a small enough* ϵ. *Then for each* $(x^0, \xi^0) \in \mathbb{R}^{2n}$ *and* $t \in [0,1]$ *we have*

$$|x^t - (x^0 + 2t\xi^0)| + |\xi^t - \xi^0| \lesssim \varepsilon$$

The proof is straightforward and is left for the reader; it is also essentially contained in [1]. This allows us to apply the main result in [8]:

Proposition 32. *Assume that* (85) *holds with a small enough* ϵ. *Then for each* $N \geq 0$ *the solution of the homogeneous problem* (84) *satisfies the following localization estimate:*

$$\|u(t)\|_{H^{N,N}_{x_0+2t\xi_0,\xi_0}} \lesssim_N \|u_0\|_{H^{N,N}_{x_0,\xi_0}} \tag{87}$$

We denote the evolution operator for (84) by $S(t,s)$. If the initial data is $u_0 = \delta_x$ then it has a decomposition of the form

$$u_0 = \sum_{\xi_0 \in \mathbb{Z}^n} u_{\xi_0}(0), \qquad \|u_{\xi_0}(0)\|_{H^{N,N}_{x,\xi^0}} \lesssim 1 \tag{88}$$

By (32), at time 1 the corresponding solutions u_{ξ_0} are concentrated close to $x + 2t\xi_0$, therefore they are spatially separated. Hence we obtain the following pointwise decay:

Corollary 33. *The kernel* $K(1,0)$ *of* $S(1,0)$ *satisfies*

$$|K(1,x,0,y)| \lesssim 1$$

The solution of the homogeneous equations (84) *satisfies*

$$\|S(1,0)u_0\|_{L^\infty} \lesssim \|u_0\|_{L^1}$$

If in addition the initial data is localized at some frequency λ, say $u_0 = S_\lambda \delta_x$ then the decomposition in (88) is restricted to the range $|\xi_0| \approx \lambda$. Then the corresponding solutions travel with speed $O(\lambda)$, and we can obtain better pointwise decay away from the propagation region:

Corollary 34. *The kernel* $K_\lambda(1,0)$ *of* $S(1,0)S_\lambda$ *satisfies*

$$|K(1,x,0,y)| \lesssim (\lambda + |x-y|)^{-N}, \qquad |x-y| \not\approx \lambda \tag{89}$$

The kernel $K_\lambda(t,s)$ *of* $S(t,s)S_\lambda$ *satisfies*

$$|K(t,x,s,y)| \lesssim \lambda^{-N}, \qquad |x-y| \approx \lambda, |t-s| \ll 1 \tag{90}$$

The next result concerns localized energy estimates.

Corollary 35. *For each ball* B_r *of radius* $r \geq 1$ *the solution* u *to* (84) *satisfies*

$$\|S(t,0)S_\lambda u_0\|_{L^2(B_r)} \lesssim \lambda^{-\frac{1}{2}} r^{\frac{1}{2}} \|u_0\|_{L^2} \tag{91}$$

Proof. We consider the wave packet decomposition for $u = S(t,0)S_\lambda u(0)$,

$$u = \sum_{(x_0,\xi_0)\in\mathbb{Z}^{2n}}^{|\xi_0|\approx\lambda} u_{x_0,\xi_0}$$

Let χ_r be a cutoff corresponding to B_r. Since $r \geq 1$ it follows that the functions $\chi_r u_{x_0,\xi_0}$ are almost orthogonal, therefore it suffices to prove the estimate for a single packet. But a single packet is concentrated near a tube of spatial size 1 which travels with speed $O(\lambda)$. This tube intersects the cylinder $[0,1]\times B_r$ over a time interval of length $\lambda^{-1}r$. The conclusion easily follows. □

To obtain any results below the unit spatial scale we slightly strengthen the condition (85) by adding a weaker pointwise bound

$$|\partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \leq c_{\alpha\beta}\langle\xi\rangle^{\frac{1}{2}}, \qquad \forall\ \alpha,\beta \tag{92}$$

This will guarantee that on a unit spatial scale the flow in (84) is a small perturbation of the flat Schrödinger flow. Then we have:

Proposition 36. *Assume that the conditions* (85) *and* (92) *hold. Then*

(i) For any $r>0$ the solution u to (84) *satisfies the localized energy estimates*

$$\|S(t,0)S_\lambda u(0)\|_{L^2(B_r)} \lesssim \lambda^{-\frac{1}{2}} r^{\frac{1}{2}} \|u_0\|_{L^2} \tag{93}$$

(ii) For each y,z with $|y-z|\approx\lambda$ we have the square function bound

$$\left\| \int_I S_\lambda S(t,s) S_\lambda (f(s)\delta_y) \right\|_{L^2} \lesssim \lambda^{-1}\|f\|_{L^2} \tag{94}$$

Proof. To prove this result it is convenient to replace the L^2 initial data space by weighted L^2 spaces.

Definition 37. *A weight $m : \mathbb{R}^{2n} \to \mathbb{R}^+$ is admissible if*

$$|m(x,\xi)/m(y,\eta)| \lesssim (1+|x-y|+|\xi-\eta|)^N$$

for some real N.

Correspondingly we define a weighted L^2 space

$$\|u\|^2_{L^2(m)} = \sum_{(x_0,\xi_0)\in\mathbb{Z}^{2n}} \|m(x_0,\xi_0)u_{x_0,\xi_0}\|^2_{H^{N,N}_{x_0,\xi_0}}$$

Given a weight m_0 at time 0 we evolve it in time by

$$m_t(x+2t\xi,\xi) = m_0(x,\xi)$$

As a consequence of Proposition 32 we obtain

Lemma 38. *Assume that* (85) *holds with a small enough ϵ. Then*

$$\|S(t,s)\|_{L^2(m_s)\to L^2(m_t)} \lesssim 1$$

Next we consider truncated solutions on a unit spatial scale. Given a unit ball B and an associated cutoff function χ we have the following weighted local energy estimates:

Lemma 39. *For any solution u to* (84) *we have*

$$\|\chi u\|_{L^2([0,1],L^2(\langle\xi\rangle^{\frac12} m_t))} + \|(i\partial_t - \Delta)\chi u\|_{L^2([0,1],L^2(\langle\xi\rangle^{-\frac12} m_t))} \lesssim \|u_0\|_{L^2(m)}$$

Proof. We begin again with a wave packet decomposition of u,

$$u = \sum_{(x_0,\xi_0)\in\mathbb{Z}^{2n}} u_{x_0,\xi_0}$$

The functions $\chi u_{x_0,\xi_0}$ are almost orthogonal in L^2 therefore the bound for χu follows. On the other hand we have

$$(i\partial_t - \Delta)\chi u = -2\nabla\chi\nabla u - \Delta\chi u + \chi a^w(t,x,D)u$$

The first two terms are estimated using the bound for χu. For the last one we note that, by (92), the operator a^w preserves the $H^{N,N}$ spaces,

$$\|a^w(t,x,D)u\|_{H^{N-1,N-1}_{x_0,\xi_0}} \lesssim \langle\xi_0\rangle^{\frac12}\|u\|_{H^{N,N}_{x_0,\xi_0}}$$

Hence we can use orthogonality again. □

Now we can conclude the proof of the Proposition. For the local energy estimate (91) we first truncate u to a unit scale. By the above lemma with $m = (1+\lambda^{-3}\langle\xi\rangle^3)(1+\lambda^3\langle\xi\rangle^{-3})$ we obtain

$$\lambda^{\frac12}\|\chi_1 u\|_{L^2(m')} + \lambda^{-\frac12}\|(i\partial_t - \Delta)\chi_1 u\|_{L^2(m')} \lesssim \|u_0\|_{L^2(m)}$$

where $m' = (1+\lambda^{-2}\langle\xi\rangle^2)(1+\lambda^2\langle\xi\rangle^{-2})$. It remains to show that

$$\lambda^{\frac12} r^{-\frac12}\|\chi_r u\|_{L^2} \lesssim \lambda^{\frac12}\|\chi_1 u\|_{L^2(m')} + \lambda^{-\frac12}\|(i\partial_t - \Delta)\chi_1 u\|_{L^2(m')}$$

Then we can localize the right hand side to the λ^{-1} time scale. On the λ^{-1} time scale we can use the Duhamel formula to further reduce the problem to a corresponding estimate for solutions to the homogeneous constant coefficient Schrödinger equation, namely:

$$\lambda^{\frac12} r^{-\frac12}\|\chi_r e^{-it\Delta} u_0\|_{L^2} \lesssim \|u_0\|_{L^2(m')}$$

After a dyadic frequency decomposition this becomes

$$\lambda^{\frac12} r^{-\frac12}\|\chi_r e^{-it\Delta} S_\lambda u_0\|_{L^2} \lesssim \|u_0\|_{L^2}$$

which is exactly the local energy estimate for the homogeneous constant coefficient Schrödinger equation.

Consider now the square function bound. For $|t-s| \ll 1$ we can use the kernel bound (90). Hence without any restriction in generality we assume that $t \in I$, $s \in J$ where I, J are intervals of size $O(1)$ with $O(1)$ separation.

Choose t_0 the center of the interval between I and J. We factor the estimate in two and prove the dual estimates

$$\left\| \int_I S(t_0, s) S_\lambda (f(s)\delta_y) \right\|_{L^2(m)} \lesssim \lambda^{-\frac{1}{2}} \|f\|_{L^2}$$

respectively

$$\|(S_\lambda S(t, t_0)u)(z)\|_{L^2} \lesssim \lambda^{-\frac{1}{2}} \|u\|_{L^2(m)}$$

where the flow invariant weight m is given by

$$m(x, \xi) = (1 + \lambda^{-1}|\xi \wedge (x - y)|)^K (1 + \lambda^{-1}|\xi \wedge (x - z)|)^{-K}$$

with K large enough. These are dual bounds therefore it suffices to prove the second one.

If χ is a smooth approximation of the characteristic function of $B(z, 1)$, then by (a slight modification of) Lemma 39 it remains to show that $v = \chi S_\lambda u$ satisfies

$$\lambda^{\frac{1}{2}} \|v(t, x)\|_{L^2(J)} \lesssim \lambda^{\frac{1}{2}} \|v\|_{L^2_t L^2(m \cdot m')} + \lambda^{-\frac{1}{2}} \|(i\partial_t - \Delta)v\|_{L^2_t L^2(m \cdot m')}$$

where the additional weight $m' = (1 + \lambda^{-2}\langle\xi\rangle^2)(1 + \lambda^2 \langle\xi\rangle^{-2})$ can be added due to the localization to frequency λ.

This estimate can be localized to the λ^{-1} timescale. In addition, since v has support in $B(z, 1)$ we can freeze $x = z$ in m and replace m by

$$\tilde{m}(\xi) = (1 + \lambda^{-1}|\xi \wedge (y - z)|)^K$$

Assuming $y - z = O(\lambda)e_1$ we get

$$\tilde{m}(\xi) = (1 + |\xi'| + \lambda^{-1}|\xi_1|)^K$$

Then the x' variable can be factored out and we are left with a bound for the one dimensional Schrödinger equation,

$$\|e^{-it\Delta} v_0(\cdot, 0)\|_{L^2(J)} \lesssim \lambda^{-\frac{1}{2}} \|v_0\|_{L^2(m')}$$

But this is exactly the one dimensional local energy estimate.

□

7. The short time structure

In this section we consider a paradifferential approximation to the magnetic Schrödinger equation (18). Precisely given a dyadic frequency λ we consider the evolution

$$iu_t - \Delta u + i(A_{<\sqrt{\lambda}} \nabla \tilde{S}_\lambda + \tilde{S}_\lambda A_{<\sqrt{\lambda}} \nabla)u = 0, \qquad u(0) = u_0 \tag{95}$$

where

$$A_{<\sqrt{\lambda}} = S_{<\sqrt{\lambda}} A$$

The multiplier $\tilde{S}_\lambda$ is added here for convenience. It guarantees that waves at frequencies away from λ evolve according to the constant coefficient Schrödinger flow, thereby strictly confining the interesting part of the evolution to frequency λ. In addition, the above expression is written in a

selfadjoint form, which guarantees that the corresponding evolution operators $S(t,s)$ are L^2 isometries.

Later we will prove that on the time scale λ^{-1} the evolution of the λ dyadic piece of a solution u to (18) is well approximated by the evolution in (95). Here we establish dispersive type estimates for (95). Our main result concerning the flow in (95) is as follows:

Proposition 40. *Let u_λ be the solution to* (95) *with initial data $u_{0,\lambda}$ localized at frequency λ. Then for any interval I of size less than λ^{-1} the following estimates hold:*

(i) the full Strichartz estimates

$$\|u_\lambda\|_{L^p(I,L^q)} \lesssim_A \|u_{0,\lambda}\|_{L^2} \tag{96}$$

(ii) the square function estimate

$$\|u_\lambda\|_{L^4_x(L^2_t(I))} \lesssim_A \lambda^{-\frac{1}{4}}\|u_{0,\lambda}\|_{L^2} \tag{97}$$

(ii) the localized energy estimate: for any ball B_r of radius $r>0$ we have

$$\|u_\lambda\|_{L^2(I\times B_r)} \lesssim_A r^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\|u_{0,\lambda}\|_{L^2} \tag{98}$$

The equation (95) is L^2 well-posed, therefore we can define the spaces $U^2_{A,\sqrt{\lambda}}L^2$, respectively $V^2_{A,\sqrt{\lambda}}L^2$.

As a consequence of the above proposition we obtain

Corollary 41. *Assume that $A\in U^2_W H^1$. Then for any interval I of length $\leq \lambda^{-1}$ and any function u_λ localized at frequency λ the following embeddings hold:*

$$\|u_\lambda\|_{L^p(I,L^q)} \lesssim_A \|u_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{99}$$

$$\|u_\lambda\|_{L^4_x(L^2_t(I))} \lesssim_A \lambda^{-\frac{1}{4}}\|u_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{100}$$

$$\|u_\lambda\|_{L^2(I\times B_r)} \lesssim_A r^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\|u_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{101}$$

The first step in the proof of Proposition 40 is to establish a wave packet parametrix and a wave packet decomposition for solutions to (95) on the λ^{-1} time scale. This is done by rescaling starting from the results in the previous section.

We begin by writing in the Weyl calculus

$$i(A_{<\sqrt{\lambda}}\nabla\tilde{S}_\lambda + \tilde{S}_\lambda A_{<\sqrt{\lambda}}\nabla) = a^w(t,x,D)$$

Then the symbol $a(t,x,\xi)$ can be expressed as a principal term plus an error,

$$a(t,x,\xi) = a_0(t,x,\xi) + a_r(t,x,\xi)$$

where the principal part a_0 is given by

$$a_0(t,x,\xi) = -2iA_{<\sqrt{\lambda}}(t,x)\cdot\xi\tilde{s}_\lambda(\xi)$$

By Sobolev embeddings we have the following pointwise bound for the truncated magnetic potential:

$$|\partial_x^\alpha A_{<\sqrt{\lambda}}(t,x)| \leq c_\alpha \lambda^{\frac{1+|\alpha|}{2}} \|A(t)\|_{H^1}$$

This yields

$$|\partial_x^\alpha \partial_\xi^\beta a_0(t,x,\xi)| \lesssim_A c_{\alpha\beta} \lambda^{\frac{3+|\alpha|}{2}-|\beta|} \tag{102}$$

In addition, by the Weyl calculus it follows that a_r is also localized at frequency λ and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a_r(t,x,\xi)| \lesssim_A c_{\alpha\beta} \lambda^{\frac{1+|\alpha|}{2}-|\beta|} \tag{103}$$

This brings us to our main integral bound for the symbol a, namely

Lemma 42. *Assume that $A \in U_W^2 H^1$ with div $A = 0$. Then the above symbol a satisfies*

$$\sup_{x,\xi} \int_0^T |\partial_x^\alpha \partial_\xi^\beta a(t,x+2t\xi,\xi)| dt \lesssim_A \left\{ \begin{array}{ll} c_\beta (T\lambda)^{\frac{1}{2}} \lambda^{-|\beta|} \log \lambda & \alpha = 0 \\ c_{\alpha\beta} (T\lambda)^{\frac{1}{2}} \lambda^{\frac{|\alpha|}{2}-|\beta|}, & |\alpha| \geq 1. \end{array} \right. \tag{104}$$

Proof. The bound for a_r follows directly from (103), therefore it remains to consider a_0. Furthermore, it suffices to consider the case $|\alpha| = 0, 1$, $\beta = 0$. Then we need to prove the bounds

$$\int_0^T |A_{<\sqrt{\lambda}}(t,x+2t\xi)| dt \lesssim T^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \ln \lambda \|\nabla A\|_{U_W^2 L^2}, \qquad |\xi| \approx \lambda$$

respectively

$$\int_0^T |\nabla A_{<\sqrt{\lambda}}(t,x+2t\xi)| dt \lesssim T^{\frac{1}{2}} \|\nabla A\|_{U_W^2 L^2}, \qquad |\xi| \approx \lambda$$

These in turn follow by dyadic summation from

$$\sup_{x,\xi} \int_0^T |(S_\mu B)(t,x+2t\xi)| dt \lesssim T^{\frac{1}{2}} \mu \lambda^{-\frac{1}{2}} \|B\|_{U_W^2 L^2}, \qquad |\xi| \approx \lambda \tag{105}$$

The line $y = x + 2t\xi$ moves through a unit spatial cube in a time λ^{-1}. But, due to the finite speed of propagation for the wave equation, see (13), the contributions from different spatial unit cubes are square summable. Hence by Cauchy-Schwartz it suffices to prove the above bound for $T \leq \lambda^{-1}$. By (12), for $T \leq \lambda^{-1}$ we have

$$\|B\|_{U_W^2(0,T;L^2)} \approx \|B\|_{U^2(0,T;L^2)},$$

therefore it is enough to prove:

$$\int_0^T |(S_\mu B)(t,x+2t\xi)| dt \lesssim (\lambda T)^{\frac{1}{2}} \mu \lambda^{-1} \|B\|_{U^2 L^2}$$

It suffices to prove the bound when B is an U^2L^2 atom. By Cauchy-Schwartz it suffices to consider a single step, which corresponds to a time independent B. Then the last bound can be rewritten in the form

$$\int_L |(S_\mu B)(x)| ds \lesssim \mu |L|^{\frac{1}{2}} \|B\|_{L^2}$$

where L is an arbitrary line segment in $\mathbb{R}^3$. We can set $\mu = 1$ by rescaling. In coordinates $x = (x_1, x')$ suppose L is contained in $\{x' = 0\}$. Then we use Sobolev embeddings in x' and Cauchy-Schwartz with respect to x_1. □

Next we consider the rescaling that preserves the flat Schrödinger flow and takes the time scale λ^{-1} to 1, namely

$$v_\lambda(x,t) = u\left(\frac{x}{\sqrt{\lambda}}, \frac{t}{\lambda}\right)$$

If u solves (95) then for v_λ we obtain the following equation:

$$iv_t - \Delta v + \lambda^{-1} a^w\left(\frac{t}{\lambda}, \frac{x}{\sqrt{\lambda}}, D\sqrt{\lambda}\right) v = 0 \tag{106}$$

However, this is not sufficient, we need to repeat the same procedure for shorter time scales. Precisely, for each $\lambda < \mu < \lambda^2$ we can rescale the μ^{-1} time scale to the unit scale by setting

$$v(x,t) = u\left(\frac{x}{\sqrt{\mu}}, \frac{t}{\mu}\right)$$

Then for v we obtain the equation

$$iv_t - \Delta v + a_\mu^w\left(t, x, D)\right) v = 0, \quad a_\mu(t,x,\xi) = \mu^{-1} a^w\left(\frac{t}{\mu}, \frac{x}{\sqrt{\mu}}, \xi\sqrt{\mu}\right) \tag{107}$$

Rescaling the bounds (102), (103) and (104) it follows that this rescaled equation belongs to the class studied in the previous section:

Lemma 43. *For $\varepsilon^{-1}\lambda \leq \mu \leq \lambda^2$ and ϵ small enough the symbol a_μ satisfies* (85) *and* (92) *on the time interval* $[0,1]$.

This allows us to apply the results in the previous section to the evolution (95). Rescaling the result in Corollary 33 we obtain short time pointwise bounds for the solution to (95):

Lemma 44. *The solution of* (95) *has the pointwise decay*

$$\|u(t)\|_{L^\infty} \lesssim |t-s|^{-\frac{n}{2}} \|u(s)\|_{L^1}, \qquad |t-s| \lesssim \epsilon\lambda^{-1} \tag{108}$$

Proof. W.a.r.g we can take $s = 0$. If $\lambda^{-2} \leq t \leq \varepsilon\lambda^{-1}$ then this follows directly from Corollary 33 applied to the equation (107) with $\mu^{-1} = t$.

The case $t < \lambda^{-2}$ needs to be considered separately. For such t we split the evolution in two parts,

$$S(t,0) = S(t,0)\tilde{\tilde{S}}_\lambda + S(t,0)(1 - \tilde{\tilde{S}}_\lambda)$$

The second part evolves according to the constant coefficient Schrödinger flow, hence it is easy to estimate. For the first part we use the rescaled parametrix in the previous section corresponding to $\mu = \lambda^{-2}$. The solution $S(t,0)\tilde{\tilde{S}}_\lambda \delta_x$ consists of a single packet on the λ^{-1} spatial scale which does not move up to time λ^{-2}. Hence we obtain

$$|S(t,0)\tilde{\tilde{S}}_\lambda \delta_x| \lesssim \lambda^3 (1+\lambda|x-y|)^{-N}, \qquad |t| \leq \lambda^{-2} \tag{109}$$

which concludes the proof. □

By [4], the Strichartz estimates in (96) are a direct consequence of (108). We continue with a decay bound away from the propagation region:

Lemma 45. *If $|t-s| \leq \varepsilon\lambda^{-1}$ then the kernel of $S(t,s)S_\lambda$ satisfies*

$$|K(t,x,s,y)| \lesssim \lambda^3 (1+\lambda|x-y| + \lambda^2|t-s|)^{-N} \tag{110}$$

whenever

$$|t-s| + \lambda^{-2} \gg \lambda^{-1}|x-y| \quad \textit{or} \quad \lambda^{-1}|x-y| + \lambda^2 \gg |t-s|$$

Proof. W.a.r.g we can take $s = 0$. If $|t-s| \leq \lambda^{-2}$ then we use (109). On the other hand if $\lambda^{-2} \leq |t-s| \leq \varepsilon\lambda^{-1}$ and then we rescale (89) applied to (107) with $\mu = t^{-1}$.

Since the input is localized at frequency λ, it follows that waves need exactly a time $\approx \lambda^{-1}|x-y| + \lambda^{-2}$ to travel from x to y.

□

Next we consider pointwise square function bounds:

Lemma 46. *The evolution $S(t,s)$ associated to* (95) *has the pointwise square function decay*

$$\left\| \int_I S(t,s)S_\lambda (f(s)\delta_y) ds(x) \right\|_{L^2_t(I)} \lesssim |x-y|^{-1} \|f\|_{L^2_t(I)}, \quad |I| \lesssim \epsilon\lambda^{-1} \tag{111}$$

Proof. If $|x-y| \gtrsim 1$ then we can use directly (110). If $|x-y| \ll 1$ then we split the integral in two parts. If $|t-s| \gg \lambda^{-1}|x-y|$ then we can still use (110). On the other hand if $|t-s| \lesssim \lambda^{-1}|x-y|$ then we rescale (94) applied to (107) with $\mu = \lambda|x-y|^{-1}$. □

We continue with the proof of (97). By the TT^* argument we need to prove the bound

$$\left\| \int_I S(t,s)S_\lambda f(s) ds \right\|_{L^4_x L^2_t} \lesssim_A \lambda^{-\frac{1}{2}} \|f\|_{L^{\frac{4}{3}}_x L^2_t} \tag{112}$$

For this we use Stein's complex interpolation theorem. Define the holomorphic family of operators

$$T_z f(t) = \int_I z(t-s)_+^{z-1} S(t,s) S_\lambda f(s) ds$$

Then we need to show that

$$\|T_1 f\|_{L^4_x L^2_t} \lesssim_A \lambda^{-\frac{1}{2}} \|f\|_{L_x^{\frac{4}{3}} L^2_t}$$

This follows by interpolation from

$$\|T_z f\|_{L^2} \lesssim \|f\|_{L^2}, \qquad \Re z = 0 \tag{113}$$

and

$$\|T_z f\|_{L^\infty_x L^2_t} \lesssim_A \lambda^{-1} \|f\|_{L^1_x L^2_t}, \qquad \Re z = 2 \tag{114}$$

For (113) we write

$$S(0,t) T_z f(t) = \int_I z(t-s)_+^{z-1} S_\lambda S(0,s) f(s) ds$$

Since $S(t,s)$ are L^2 isometries it suffices to prove that

$$\| \int_I z(t-s)_+^{z-1} f(s) ds \|_{L^2} \lesssim \|f\|_{L^2}$$

which is straightforward by Plancherel's theorem since the Fourier transform of zt_+^{z-1} is $\Gamma(z+1)(\tau + i0)^{-z}$ which is bounded.

On the other hand the bound (114) is equivalent to

$$\|T_z(f\delta_y)(x)\|_{L^2_t} \lesssim_A \lambda^{-1} \|f\|_{L^2_t}$$

which we can rewrite in the form

$$\left\| \int_I (t-s)^{1+i\sigma} S_{\sqrt{\lambda}}(t,s) S_\lambda (f(s)\delta_y) ds(x) \right\|_{L^2_t(I)} \lesssim_A \lambda^{-1} \|f\|_{L^2_t(I)}$$

Restricting $t-s$ to the range $|t-s| \lesssim \lambda^{-1}|x-y| + \lambda^{-2}$ this is a consequence of (111). On the other hand for larger $t-s$ we can use directly the pointwise bound (110).

The last step of the proof of Proposition 40 is the localized energy estimate (98). This follows directly by rescaling from (93) applied to the equation (106).

8. Short range bilinear and trilinear estimates

We first consider L^2 bilinear product estimates where one factor solves the wave equation and the other solves the Schrödinger equation.

Proposition 47. *Assume that $A \in U^2_W H^1$, $1 \leq \mu \lesssim \lambda$ and $|I| \leq \lambda^{-1}$. Then the following bilinear L^2 estimates hold:*

$$\|S_\mu(B_\lambda u_\lambda)\|_{L^2(I\times\mathbb{R}^3)} \lesssim_A \mu^{\frac{1}{2}} \|B_\lambda\|_{U^2_W L^2} \|u_\lambda\|_{U^2_{A,\sqrt{\lambda}} L^2}, \tag{115}$$

$$\|B_\lambda u_\mu\|_{L^2(I\times\mathbb{R}^3)} \lesssim_A \mu^{\frac{1}{2}} \|B_\lambda\|_{U^2_W L^2} \|u_\mu\|_{U^2_{A,\sqrt{\mu}} L^2} \tag{116}$$

$$\|B_\mu u_\lambda\|_{L^2(I\times\mathbb{R}^3)} \lesssim_A \mu\lambda^{-\frac{1}{2}} \|B_\mu\|_{U^2_W L^2} \|v_\lambda\|_{U^2_{A,\sqrt{\lambda}} L^2}, \tag{117}$$

We remark that the constants in (117) are optimal, and in effect as a consequence of the results in the last section (117) can be extended almost up to time 1. On the other hand the constants in (115), (116) are not optimal, but this is not so important because this corresponds to the non-resonant case in the trilinear estimates.

Proof. For (115) it suffices to use Bernstein's inequality and the Strichartz estimates,

$$\begin{aligned}\|S_\mu(B_\lambda u_\lambda)\|_{L^2} &\lesssim \mu^{\frac{1}{2}}\|B_\lambda u_\lambda\|_{L^2_t L^{\frac{3}{2}}_x}\\ &\lesssim \mu^{\frac{1}{2}}\|B_\lambda\|_{L^\infty L^2}\|u_\lambda\|_{L^2_t L^6_x}\\ &\lesssim_A \mu^{\frac{1}{2}}\|B_\lambda\|_{U^2_W L^2}\|u_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2}\end{aligned}$$

A similar argument applies for (116).

It remains to prove (117). By (12) we replace the $U^2_W L^2$ space by the $U^2 L^2$ space on a short time scale. Hence we can rewrite (117) in the form

$$\|B_\mu u_\lambda\|_{L^2(I\times\mathbb{R}^3)} \lesssim_A \mu\lambda^{-\frac{1}{2}}\|B_\mu\|_{U^2L^2}\|v_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{118}$$

Due to the atomic structure of the U^2 spaces it suffices to prove the above bound in the special case when both B_μ and u_λ solve the corresponding homogeneous equations $\partial_t B_\mu = 0$, respectively (95).

We consider a partition of unit on the μ^{-1} scale

$$1 = \sum_{x_0\in\mu^{-1}\mathbb{Z}^3}\phi^2_{x_0}(x)$$

and use the localized energy estimates (101) for u_λ with $r=\mu^{-1}$:

$$\begin{aligned}\|B_\mu u_\lambda\|^2_{L^2(I\times\mathbb{R}^3)} &\approx \sum_{x_0\in\mu^{-1}\mathbb{Z}^3}\|\phi^2_{x_0}B_\mu u_\lambda\|^2_{L^2}\\ &\lesssim \sum_{x_0\in\mu^{-1}\mathbb{Z}^3}\|\phi_{x_0}B_\mu\|^2_{L^\infty}\|\phi_{x_0}u_\lambda\|^2_{L^2}\\ &\lesssim_A \lambda^{-1}\mu^{-1}\|u_\lambda\|^2_{U^2_{A,\sqrt{\lambda}}L^2}\sum_{x_0\in\mu^{-1}\mathbb{Z}^3}\|\phi_{x_0}B_\mu\|^2_{L^\infty}\\ &\lesssim \lambda^{-1}\mu^2\|B_\mu\|_{U^2L^2}\|v_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2}\end{aligned}$$

□

Next we turn our attention to trilinear estimates. We begin with the easier case of three U^2 type spaces

Proposition 48. *a) If $|I|\leq\lambda^{-1}$, $\mu\lesssim\lambda$ and B_μ, u_λ, v_λ are localized at frequency μ, λ, respectively λ then*

$$\left|\int_I\int_{\mathbb{R}^3}B_\mu u_\lambda\bar{v}_\lambda dxdt\right| \lesssim_A \frac{\min(\mu,\lambda^{\frac{1}{2}})}{\lambda}\|B_\mu\|_{U^2_W L^2}\|u_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2}\|v_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{119}$$

b) If $|I| = \lambda^{-1}$, $\mu \ll \lambda$ and B_λ, u_μ, v_λ are localized at frequency λ, μ, respectively λ then

$$\left|\int_I \int_{\mathbb{R}^3} B_\lambda u_\mu \bar{v}_\lambda dxdt\right| \lesssim_A \mu^{\frac12}\lambda^{-1}\|B_\lambda\|_{U^2_W L^2}\|v_\mu\|_{U^2_{A,\sqrt{\mu}}L^2}\|w_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2} \tag{120}$$

Proof. a) If $\mu < \lambda^{\frac12}$ then the conclusion follows directly from (117) since

$$\left|\int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dxdt\right| \lesssim |I|^{\frac12}\|B_\mu u_\lambda\|_{L^2}\|v_\lambda\|_{L^\infty L^2}$$

If $\mu > \lambda^{\frac12}$ then we use (12) to replace $U^2_W L^2$ by $U^2L^2 \subset L^2_x L^\infty_t$. Then we estimate

$$\left|\int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dxdt\right| \lesssim_A \|B_\mu\|_{L^2_x L^\infty_t}\|u_\lambda\|_{L^4_x L^2_t}\|v_\lambda\|_{L^4_x L^2_t}$$

and use the square function bounds (97) for the last two factors.

b) In the Fourier space we obtain nontrivial contributions when either all three time frequencies are $\ll \lambda^2$ or when at least two of them are $\gtrsim \lambda^2$. More precisely, using smooth time multiplier cutoffs we can write

$$\begin{aligned}\int_I \int_{\mathbb{R}^3} B_\lambda u_\mu \bar{v}_\lambda dxdt = &\int_I \int_{\mathbb{R}^3} \chi_{\{|D_t|>\lambda^2/32\}} B_\lambda u_\mu \bar{v}_\lambda dxdt\\ &+ \int_I \int_{\mathbb{R}^3} \chi_{\{|D_t|<\lambda^2/32\}} B_\lambda \chi_{\{|D_t|>\lambda^2/32\}} u_\mu \bar{v}_\lambda dxdt\\ &+ \int_I \int_{\mathbb{R}^3} \chi_{\{|D_t|<\lambda^2/32\}} B_\lambda \chi_{\{|D_t|<\lambda^2/32\}} u_\mu \chi_{\{|D_t|<\lambda^2/16\}} \bar{v}_\lambda dxdt\end{aligned}$$

Since the wave equation has constant coefficients, for the first term we can bound the first factor in L^2,

$$\|\chi_{\{|D_t|>\lambda^2/32\}} B_\lambda\|_{L^2} \lesssim \lambda^{-1}\|B_\lambda\|_{U^2_W L^2}$$

On the other hand for the remaining product we use the energy estimate for v_λ and the L^2L^∞ bound for u_μ.

We argue in a similar manner for the other two terms. The bilinear expressions

$$\chi_{\{|D_t|<\lambda^2/32\}} B_\lambda \chi_{\{|D_t|<\lambda^2/32\}} u_\mu, \qquad S_\mu(\chi_{\{|D_t|<\lambda^2/32\}} B_\lambda \bar{v}_\lambda)$$

can be estimated in L^2 using (116), respectively (115). Hence it remains to bound in L^2 the high modulation factors:

Lemma 49. *We have*

$$\|\chi_{\{|D_t|<\lambda^2/16\}} v_\lambda\|_{L^2} \lesssim_A \lambda^{-1}\|v_\lambda\|_{U^2_{A,\sqrt{\lambda}}L^2}$$

respectively

$$\|\chi_{\{|D_t|>\lambda^2/32\}} u_\mu\|_{L^2} \lesssim_A \lambda^{-1}\|u_\mu\|_{U^2_{A,\sqrt{\mu}}L^2}$$

Proof. In the case $A = 0$ both bounds are trivial, the difficulty is to accommodate the unbounded term involving A. We consider the first bound only, as the argument for the second is similar.

Without any restriction in generality we can take v_λ to be an U^2 atom. The kernels of the operators $\chi_{\{|D_t|<\lambda^2/16\}}$, respectively $\chi_{\{|D_t|>\lambda^2/32\}}$ decay rapidly on the λ^{-2} time scale. Then it suffices to prove the estimate in two cases:

(i) v_λ is supported in a λ^{-2} time interval (this corresponds to steps of length λ^{-2} and shorter). Then the bound follows directly from the energy estimates and Holder's inequality.

(ii) v_λ solves the homogeneous equation (95) on the time interval I with $\lambda^{-2} \leq |I| \leq \lambda^{-1}$ (this corresponds to steps of length λ^{-2} and longer). Then we can use the bound (117) to estimate

$$\|(i\partial_t - \Delta)v_\lambda\|_{L^2(I)} = \|A_{<\sqrt{\lambda}} \nabla \tilde{S}_\lambda v_\lambda\|_{L^2(I)} \lesssim_A \ln\lambda \; \lambda^{\frac{1}{2}} \|v_\lambda\|_{L^\infty L^2}$$

Hence with $I = [t_0, t_1]$ we can write

$$(i\partial_t - \Delta)(\chi_I v_\lambda) = i v_\lambda(t_0)\delta_{t=t_0} - i v_\lambda(t_1)\delta_{t=t_1} + f_\lambda$$

where χ_I is the characteristic function of I and

$$\|f_\lambda\|_{L^2} \lesssim_A \ln\lambda \; \lambda^{\frac{1}{2}} \|v_\lambda\|_{L^\infty L^2}$$

Hence working with the constant coefficient Schrödinger equation we obtain

$$\begin{aligned}\|\chi_{\{|D_t|<\lambda^2/2\}}\chi_I v_\lambda\|_{L^2} \lesssim_A & \lambda^{-1}(\|v_\lambda(t_0)\|_{L^2} + \|v_\lambda(t_1)\|_{L^2}) + \lambda^{-2}\|f\|_{L^2} \\ \lesssim_A & \lambda^{-1}\|v_\lambda\|_{L^\infty L^2}\end{aligned}$$

which is exactly what we need.

□

□

Finally we turn our attention to the case when one of the three U^2 spaces is replaced by a V^2 space:

Proposition 50. *a) Let $|I| \leq \lambda^{-1}$, $\mu \lesssim \lambda$ and B_μ, u_λ, v_λ localized at frequency μ, λ, respectively λ. Then*

$$\left| \int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dx dt \right| \lesssim_A \frac{(\lambda|I|)^{\frac{1}{2}}\mu}{\lambda} \|B_\mu\|_{U^2_W L^2} \|u_\lambda\|_{U^2_{A,\sqrt{\lambda}} L^2} \|v_\lambda\|_{V^2_{A,\sqrt{\lambda}} L^2} \tag{121}$$

If in addition $\lambda^{\frac{1}{2}} \ll \mu \lesssim \lambda$ then

$$\left| \int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dx dt \right| \lesssim_A \frac{\ln\left(\frac{\mu}{\sqrt{\lambda}}\right)}{\sqrt{\lambda}} \|B_\mu\|_{U^2_W L^2} \|u_\lambda\|_{U^2_{A,\sqrt{\lambda}} L^2} \|v_\lambda\|_{V^2_{A,\sqrt{\lambda}} L^2} \tag{122}$$

b) If $|I| = \lambda^{-1}$, $\mu \ll \lambda$ and B_λ, u_μ, v_λ are localized at frequency λ, μ, respectively λ then

$$\left| \int_I \int_{\mathbb{R}^3} B_\lambda u_\mu \bar{v}_\lambda dxdt \right| \lesssim_A \frac{\mu^{\frac{1}{2}} \ln \lambda}{\lambda} \|B_\lambda\|_{U^2_W L^2} \|u_\mu\|_{U^2_{A,\sqrt{\mu}} L^2} \|v_\lambda\|_{V^2_{A,\sqrt{\lambda}} L^2} \tag{123}$$

Proof. a) Using the bilinear L^2 bound (117) for the product of the first two factors we obtain

$$\left| \int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dxdt \right| \lesssim_A \frac{(\lambda |I|)^{\frac{1}{2}} \mu}{\lambda} \|B_\mu\|_{U^2_W L^2} \|u_\lambda\|_{U^2_{A,\sqrt{\lambda}} L^2} \|v_\lambda\|_{L^\infty L^2}$$

Then (121) follows due to the trivial embedding $V^2_{A,\sqrt{\lambda}} L^2 \subset L^\infty_t L^2_x$.

On the other hand the LHS of (122) can be estimated either as above or as in (119). Then (122) follows from the decomposition

$$V^2_{A,\sqrt{\lambda}} L^2 \subset \ln \left(\frac{\mu}{\sqrt{\lambda}} \right) U^2_{A,\sqrt{\lambda}} L^2 + \left(\frac{\mu}{\sqrt{\lambda}} \right)^{-1} L^2$$

We can factor out the (95) flow by pulling functions back to time 0 along the flow. Then the above relation becomes

$$V^2 L^2 \subset \ln \sigma \; U^2 L^2 + \sigma^{-1} \; L^\infty L^2, \qquad \sigma \gg 1$$

This in turn is true due to Lemma 8.

b) This follows from a similar argument to the one above and by using (116) and (120).

□

9. The short time paradifferential calculus

In this section we prove that, given a dyadic frequency λ, the evolution of the λ dyadic piece of a solution u to (18) is well approximated by the evolution of the paradifferential equation (95) on time intervals of size λ^{-1}. We also introduce different paradifferential truncations

$$iu_t - \Delta u + i(A_{<\nu} \nabla \tilde{S}_\lambda + \tilde{S}_\lambda A_{<\nu} \nabla) u = 0, \qquad u(0) = u_0 \tag{124}$$

and show that they all generate equivalent spaces. The spaces associated to (124) are denoted by $U^2_{A,\nu,\lambda} L^2$, $U^2_{A,\nu,\lambda} L^2$, respectively $U^2_{A,\nu,\lambda} L^2$. We refer to the above evolution as the $(A_{<\nu}, \lambda)$ flow.

A special case of the above equation is when $A_{<\nu}$ is replaced by $A_{\ll\lambda}$. We refer to that as the $(A_{\ll\lambda}, \lambda)$ flow. By a slight abuse of notation we denote the corresponding spaces by $U^2_{A,\lambda,\lambda} L^2$, etc.

Proposition 51. *a) For any interval I with $|I| \leq \lambda^{-1}$ the solution u to (18) satisfies*

$$\|S_\lambda u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)} \lesssim_A \|u_0\|_{L^2} \tag{125}$$

b) In addition, for any $\sqrt{\lambda} < \nu \ll \lambda$ we have

$$\|u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)} \approx_A \|u\|_{U^2_{A,\nu,\lambda}(I;L^2)} \tag{126}$$

The bound (125) transfers easily to U^2 spaces:

Corollary 52. *For any interval I with $|I| \leq \lambda^{-1}$ we have*

$$\|S_\lambda u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)} \lesssim_A \|u\|_{U^2_A(I;L^2)} \tag{127}$$

Combining this with (99) we immediately obtain the bounds (20) in part (i) of Theorem 9:

Corollary 53. *The solution u to the homogeneous magnetic Schrödinger equation* (18) *satisfies the Strichartz estimates* (20)*.*

The rest of the section is dedicated to the proof of Proposition 51.

Proof. a) From the equation (18) we obtain the following equation for $S_\lambda u$,

$$\left(i\partial_t - \Delta + iA_{<\sqrt{\lambda}}\nabla\tilde{S}_\lambda + i\tilde{S}_\lambda A_{<\sqrt{\lambda}}\nabla\right) S_\lambda u = f_\lambda$$

where

$$f_\lambda = S_\lambda(2iA_{>\sqrt{\lambda}}\nabla u + A^2 u) + i[S_\lambda, A_{<\sqrt{\lambda}}]\tilde{S}_\lambda \nabla u$$

Then we have

$$\|S_\lambda u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)} \lesssim \|u_0\|_{L^2} + \|f_\lambda\|_{DU^2_{A,\sqrt{\lambda}}(I;L^2)}$$

The estimate (125) follows if we establish that the inhomogeneous terms f_λ are uniformly small,

$$\|f_\lambda\|_{DU^2_{A,\sqrt{\lambda}}(I;L^2)} \lesssim_A (\lambda|I|)^\delta \sup_{\lambda'} \|S_{\lambda'} u\|_{U^2_{A,\sqrt{\lambda'}}(I;L^2)}$$

We consider the terms in f_λ. For the first term by duality we need to prove that

$$\left|\int_I \int_{\mathbb{R}^3} A_{>\sqrt{\lambda}}\nabla u S_\lambda \bar{v}\, dx dt\right| \lesssim_A (\lambda|I|)^\delta \|v\|_{V^2_{A,\sqrt{\lambda}}L^2} \sup_{\lambda'} \|S_{\lambda'} u\|_{U^2_{A,\sqrt{\lambda'}}(I;L^2)}$$

We take a Littlewood-Paley decomposition of the first two factors and estimate each dyadic piece. There are several cases to consider for the integrand:

a) $S_\mu A \nabla S_\lambda u S_\lambda \bar{v}$ with $\sqrt{\lambda} \leq \mu \leq \lambda$. Using (121) yields a constant

$$|I|^{\frac{1}{2}}\sqrt{\lambda} = (\lambda|I|)^{\frac{1}{4}}(\mu^2|I|)^{\frac{1}{4}}\left(\frac{\lambda}{\mu^2}\right)^{\frac{1}{4}}$$

which is favorable if $|I| \leq \mu^{-2}$. On the other hand using (122) yields a constant

$$\ln\left(\frac{\mu}{\sqrt{\lambda}}\right)\mu^{-1}\sqrt{\lambda} = \ln\left(\frac{\mu}{\sqrt{\lambda}}\right)\left(\frac{\lambda}{\mu^2}\right)^{\frac{1}{4}}(|I|\lambda)^{\frac{1}{4}}(|I|\mu^2)^{-\frac{1}{4}}$$

which is favorable if $|I| \geq \mu^{-2}$.

b) $S_\lambda A \nabla S_\mu u S_\lambda \bar{v}$ with $\mu \ll \lambda$. Then using (123) yields a constant

$$\mu^{\frac{3}{2}}\lambda^{-2}\ln\lambda \leq \lambda^{-\frac{1}{2}}\ln\lambda$$

and a power of $|I|$ can be easily gained as in Remark 10.

c) $S_\nu A \nabla S_\nu u S_\lambda \bar{v}$ with $\nu \gg \lambda$. Then we can use (123) but only on ν^{-1} time intervals. We obtain a constant

$$\lambda^{\frac{1}{2}} \nu^{-1} \ln \nu$$

and again a power of I is gained as in Remark 10.

For the second term in f_λ by duality we need to prove that

$$\left| \int_I \int_{\mathbb{R}^3} A^2 u S_\lambda \bar{v} dx dt \right| \lesssim_A (\lambda |I|)^\delta \|v\|_{V^2_{A,\sqrt{\lambda}} L^2} \sup_\lambda \|S_\lambda u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)}$$

We consider the corresponding dyadic pieces

$$\begin{aligned} \left| \int_I \int_{\mathbb{R}^3} S_{\lambda_1} A S_{\lambda_2} A S_{\lambda_3} u S_\lambda \bar{v} dx dt \right| & \\ & \leq (\lambda |I|)^{\frac{5}{12}} \|S_{\lambda_1} A\|_{L^6 L^3} \|S_{\lambda_2} A\|_{L^6 L^3} \|S_{\lambda_3} u\|_{L^\infty L^2} \|S_\lambda v\|_{L^4 L^3} \\ & \lesssim_A (\lambda |I|)^{\frac{5}{12}} \lambda_1^{-\frac{2}{3}} \lambda_2^{-\frac{2}{3}} \|A\|^2_{U^2_W H^1} \|S_{\lambda_3} u\|_{L^\infty L^2} \|v\|_{V^2_{A,\sqrt{\lambda}} L^2} \end{aligned}$$

where for v we have used the short time Strichartz estimates. The summation with respect to λ_1 and λ_2 is trivial. So is the summation with respect to λ_3 since the integral is zero unless either $\lambda_3 = \lambda$ or $\lambda_3 \leq \max\{\lambda_1, \lambda_2\}$.

Finally for the commutator term in f_λ we have the bound

$$\begin{aligned} \|[S_\lambda, A_{<\sqrt{\lambda}}] \nabla \tilde{S}_\lambda u\|_{L^1 L^2} & \lesssim \lambda^{-1} (\lambda |I|) \|[S_\lambda, A_{<\sqrt{\lambda}}] \nabla \tilde{S}_\lambda u\|_{L^\infty L^2} \\ & \lesssim \lambda^{-1} (\lambda |I|) \|\nabla A_{<\sqrt{\lambda}}\|_{L^\infty} \|\tilde{S}_\lambda u\|_{L^\infty L^2} \\ & \lesssim \lambda^{-\frac{1}{4}} (\lambda |I|) \|A\|_{L^\infty H^1} \|\tilde{S}_\lambda u\|_{L^\infty L^2} \end{aligned}$$

which again suffices by duality.

b) By the same argument as in part (a) we obtain

$$\|A_{<\sqrt{\lambda}} \nabla u - A_{<\nu} \nabla u\|_{DU^2_{A,\sqrt{\lambda}}(I;L^2)} \lesssim_A (\lambda |I|)^\delta \|u\|_{U^2_{A,\sqrt{\lambda}}(I;L^2)}$$

which shows that the two flows are close. Then (126) follows due to Lemma 3. $\square$

10. Generalized wave packet decompositions and long range trilinear estimates

Denote by $C_1(\mu, \lambda, |I|)$ the best constant in the estimate

$$\left| \int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dx dt \right| \leq C_1(\mu, \lambda, |I|) \|B_\mu\|_{U^2_W L^2} \|u_\lambda\|_{U^2_{A,\lambda,\lambda} L^2} \|v_\lambda\|_{U^2_{A,\lambda,\lambda} L^2}$$

with B_μ, u_λ and v_λ localized at frequencies μ, λ, respectively λ. Similarly, let $C_2(\mu, \lambda, |I|)$ be the best constant in the estimate

$$\left| \int_I \int_{\mathbb{R}^3} B_\lambda u_\mu \bar{v}_\lambda dx dt \right| \leq C_2(\mu, \lambda, |I|) \|B_\lambda\|_{U^2_W L^2} \|u_\mu\|_{U^2_{A,\mu,\mu} L^2} \|w_\lambda\|_{U^2_{A,\lambda,\lambda} L^2}$$

with B_λ, u_μ and v_λ localized at frequencies λ, μ, respectively λ.

As a consequence of Proposition 48 we have

$$C_1(\mu,\lambda,\lambda^{-1}) \lesssim_A \lambda^{-\frac{1}{2}} \min\{\mu\lambda^{-\frac{1}{2}},1\}$$

A trivial summation shows that for larger time intervals we have

$$C_1(\mu,\lambda,|I|) \lesssim_A \lambda^{-\frac{1}{2}} \min\{\mu\lambda^{-\frac{1}{2}},1\}(1+\lambda|I|) \tag{128}$$

We seek to iteratively improve this to

$$C_1(\mu,\lambda,|I|) \lesssim_A \lambda^{-\frac{1}{2}} \min\{\mu\lambda^{-\frac{1}{2}},1\}(1+\lambda|I|)^{\frac{1}{2}} \tag{129}$$

for intervals I almost up to length 1. To achieve this we iteratively produce an increasing sequence of times T for which (129) holds for $|I| \leq T$. At the same time we seek to improve $C_2(\mu,\lambda,|I|)$ in a similar manner, as well as extend the time for which the local energy and local Strichartz estimates hold. More precisely, we consider a set of properties as follows:

(i) Paradifferential approximation of the flow:

$$\|S_\lambda u\|_{U^2_{A,\lambda,\lambda}(I;L^2)} \lesssim_A \|u\|_{U^2_A(I,L^2)} \tag{130}$$

(ii) Trilinear bounds:

$$C_1(\mu,\lambda,|I|) \lesssim_A \lambda^{-\frac{1}{2}} \min\{\mu\lambda^{-\frac{1}{2}},1\}(1+\lambda|I|)^{\frac{1}{2}} \tag{131}$$

$$C_2(\mu,\lambda,|I|) \lesssim_A \mu^{\frac{1}{2}}\lambda^{-1}(1+\lambda|I|)^{\frac{1}{2}} \tag{132}$$

(iii) Local energy and local Strichartz estimates for each cube Q of size 1 and each function u_λ localized at frequency λ:

$$\|u_\lambda\|_{L^2(I;L^2(Q))} \lesssim_A \lambda^{-\frac{1}{2}}\|u_\lambda\|_{U^2_{A,\lambda,\lambda}L^2}, \tag{133}$$

$$\|u_\lambda\|_{L^2(I;L^6(Q))} \lesssim_A \|u_\lambda\|_{U^2_{A,\lambda,\lambda}L^2}. \tag{134}$$

So far, by Propositions 40,51,48 we know that the above estimates (130)-(134) hold if $|I| \leq \lambda^{-1}$. On the other hand, in order to prove Theorem 9 we need to know that (130)-(134) hold if $|I| \leq \lambda^{-\epsilon}$ for ε arbitrarily small (see also Remark 10). This is accomplished in the next result.

Proposition 54. *Let $0 < \alpha \leq 1$. Assume that the estimates (130)-(134) hold for $|I| < \lambda^{-\alpha}$. Then (130)-(134) hold for $|I| < \lambda^{-\beta}$ for each $\beta > \frac{3}{4}\alpha$.*

Proof. We first improve the time range of the paradifferential calculus:

Lemma 55. *a) For each frequency λ we have*

$$\|S_\lambda u\|_{U^2_{A,\lambda,\lambda}(I,L^2)} \lesssim_A \|u\|_{U^2_A(I,L^2)}, \qquad |I| \leq \lambda^{-\frac{\alpha}{2}}(\log\lambda)^{-\frac{3}{2}}$$

b) For each frequency $\lambda^{1-\frac{\alpha}{2}}\log\lambda < \nu \ll \lambda$ we have

$$\|u\|_{U^2_{A,\nu,\lambda}(I,L^2)} \approx_A \|u\|_{U^2_{A,\lambda,\lambda}(I,L^2)}, \qquad |I| \leq T(\lambda,\nu) = \nu\lambda^{-1-\frac{\alpha}{2}}(\log\lambda)^{-1}$$

Proof. a) We observe that (131), (132), (133) for $|I| = \lambda^{-\alpha}$ trivially lead to bounds for longer time,

$$C_1(\mu,\lambda,|I|) \lesssim_A \lambda^{-\frac{1}{2}} \min\{\mu\lambda^{-\frac{1}{2}},1\}(1+\lambda|I|)^{\frac{1}{2}}(1+\lambda^\alpha|I|)^{\frac{1}{2}} \tag{135}$$

$$C_2(\mu,\lambda,|I|) \lesssim_A \mu^{\frac{1}{2}}\lambda^{-1}(1+\lambda|I|)^{\frac{1}{2}}(1+\lambda^\alpha|I|)^{\frac{1}{2}} \tag{136}$$

$$\|u_\lambda\|_{L^2(I,L^6(Q))} \lesssim_A (1+\lambda^\alpha|I|)^{\frac{1}{2}}\|u_\lambda\|_{U^2_{A,\lambda,\lambda}L^2} \tag{137}$$

Furthermore, due to (130), we obtain the same constant in the trilinear estimates when we replace u_λ by $S_\lambda u$ and $\|u_\lambda\|_{U^2_{A,\lambda,\lambda}L^2}$ by $\|u\|_{U^2_A L^2}$. Also by the argument in Lemma 8, we can also replace one of the U^2 norms with a V^2 norm at the expense of an additional $\ln\lambda$ loss.

The rest of the proof is similar to the proof of Proposition 53. For u solving (18) we write

$$\left(i\partial_t - \Delta + iA_{\ll\lambda}\nabla\tilde{S}_\lambda + i\tilde{S}_\lambda A_{\ll\lambda}\nabla\right) S_\lambda u = f_\lambda$$

where

$$f_\lambda = S_\lambda(2iA_{\gtrsim\lambda}\nabla u + A^2 u) + i[S_\lambda, A_{\ll\lambda}]\tilde{S}_\lambda\nabla u$$

Hence it suffices to prove that

$$\|f_\lambda\|_{DU^2_{A,\lambda,\lambda}(I;L^2)} \lesssim_A \|u\|_{U^2_A(I,L^2)}$$

We use duality and consider each term in f_λ. For the first one we need to show that

$$\left|\int_I\int_{\mathbb{R}^3} A_{\gtrsim\lambda}\nabla u S_\lambda\bar{v}\,dxdt\right| \lesssim_A \|A\|_{U^2_W H^1}\|u\|_{U^2_A(I,L^2)}\|v\|_{V^2_{A,\lambda,\lambda}L^2}$$

After a Littlewood-Paley decomposition of the first two factors we need to consider the following three cases for the integrand:

(i) $S_\lambda A\nabla S_\lambda u S_\lambda\bar{v}$. Then we use (135) to obtain a constant

$$\log\lambda\ \lambda^{\frac{1}{2}}(\lambda|I|)^{\frac{1}{2}}(\lambda^\alpha|I|)^{\frac{1}{2}}\lambda^{-1} = \log\lambda\ \lambda^{\frac{\alpha}{2}}|I|$$

which is satisfactory given the range for I.

(ii) $S_\lambda A\nabla S_\mu u S_\lambda\bar{v}$, $\mu \ll \lambda$. Then we use (136) to obtain a constant

$$\mu^{\frac{3}{2}}\lambda^{-1}(\lambda|I|)^{\frac{1}{2}}(\lambda^\alpha|I|)^{\frac{1}{2}}\lambda^{-1} \leq \lambda^{\frac{\alpha}{2}}|I|$$

which is much better than we need.

(iii) $S_\nu A\nabla S_\nu u S_\lambda\bar{v}$, $\lambda \ll \nu$. Then we use (136) to obtain a constant

$$\lambda^{\frac{3}{2}}\nu^{-1}(\nu|I|)^{\frac{1}{2}}(\nu^\alpha|I|)^{\frac{1}{2}}\nu^{-1} \leq \lambda^{\frac{\alpha}{2}}|I|$$

For the second term in f_λ by duality we need to prove that

$$\left|\int_I\int_{\mathbb{R}^3} ABuS_\lambda\bar{v}\,dxdt\right| \lesssim_A \|A\|_{U^2_W H^1}\|B\|_{U^2_W H^1}\|u\|_{U^2_A(I,L^2)}\|v\|_{V^2_{A,\lambda,\lambda}L^2}$$

Due to the finite speed of propagation for the wave equation, see (13), it suffices to consider the case when A and B are supported in a unit cube.

Then we bound A and B in $L^\infty L^6$, u in $L^2 L^6$ as in (137), and v in $L^\infty L^2$ and use Holder's inequality with respect to time. This yields the same constant $\lambda^{\frac{\alpha}{2}}|I|$ as above.

Finally we consider the commutator term in f_λ. This can be represented in the form of a rapidly convergent series of the form

$$[S_\lambda, A_{\ll\lambda}]\nabla \tilde{S}_\lambda u = \sum_j S_\lambda^{1j}(\nabla A_{\ll\lambda} S_\lambda^{2j} u)$$

where $S_\lambda^{1j}, S_\lambda^{2j}$ are operators similar to S_λ.

Then by duality it suffices to prove that

$$\left| \int_I \int_{\mathbb{R}^2} \nabla A_{\ll\lambda} \nabla S_\lambda u S_\lambda \bar{v} dx dt \right| \lesssim_A (\log\lambda)^{\frac{3}{2}} \lambda^{\frac{\alpha}{2}} |I| \|\nabla A\|_{U^2_W L^2} \|u\|_{U^2_A L^2} \|v\|_{V^2_{A,\lambda,\lambda} L^2}$$

For this it suffices to consider a Littlewood-Paley decomposition of $A_{\ll\lambda}$ and to apply (135) for each dyadic piece.

b) By virtue of Lemma 3 it suffices to show that

$$\|A_{\ll\lambda} \nabla \tilde{S}_\lambda u - A_{<\nu} \nabla \tilde{S}_\lambda u\|_{DU^2_{A,\lambda,\lambda}} \lesssim_A \|u\|_{U^2_{A,\lambda,\lambda}}$$

and the similar bound for $\tilde{S}_\lambda A_{\ll\lambda}\nabla - \tilde{S}_\lambda A_{<\nu}\nabla$. By duality this becomes

$$\left| \int_I \int_{\mathbb{R}^3} \sum_{\mu=\nu}^{\lambda} S_\mu A \nabla \tilde{S}_\lambda u \bar{v} dx dt \right| \lesssim_A \|A\|_{U^2_A H^1} \|u\|_{U^2_{A,\lambda,\lambda}} \|v\|_{V^2_{A,\lambda,\lambda}}$$

Indeed, for $|I| > \lambda^\alpha$ the estimate (135) yields a constant

$$\log\lambda\ \lambda^{\frac{1}{2}} (\lambda|I|)^{\frac{1}{2}} (\lambda^\alpha |I|)^{\frac{1}{2}} \nu^{-1} = \log\lambda\ \nu^{-1} \lambda^{\frac{\alpha}{2}+1} |I|$$

which leads to the restriction

$$|I| \le T(\lambda,\nu) = (\log\lambda)^{-1} \nu \lambda^{-\frac{\alpha}{2}-1}$$

We observe that this is useful only if it provides information on time intervals with $|I| > \lambda^{-\alpha}$. This leads to the condition

$$\nu > \lambda^{1-\frac{\alpha}{2}} \log\lambda.$$

□

Next we consider the (124) evolution and we construct a generalized wave packet structure for the flow. The frequency scale is $\delta\xi = \nu$ and the time scale is $T(\lambda,\nu)$ therefore it is natural to define the spatial scale by $\delta x = \nu T(\lambda,\nu)$, as in the case of the flat flow.

We first partition the initial data. Let ϕ be a smooth unit bump function in $\mathbb{R}^3 \times \mathbb{R}^3$ so that

$$\sum_{k,j\in\mathbb{Z}^n} \phi(x-k, \xi-j) = 1$$

Denote

$$\phi^\nu_{x_0,\xi_0}(x,\xi) = \phi\left(\frac{x-x_0}{\nu T(\lambda,\nu)}, \frac{\xi-\xi_0}{\nu}\right)$$

where

$$(x_0,\xi_0)\in \mathbb{Z}_\nu^{2\times 3} = (\nu T(\lambda,\nu)\mathbb{Z})^3\times(\nu\mathbb{Z})^3$$

Then consider an almost orthogonal decomposition of the initial data

$$u_0 = \sum_{(x_0,\xi_0)\in\mathbb{Z}_\nu^{2\times 3}} \phi^\nu_{x_0,\xi_0}(x,D)u_0$$

We denote the corresponding solutions to (124) by u_{x_0,ξ_0}, and we call them generalized wave packets. To measure their evolution we consider the family of operators

$$L_{x_0,\xi_0} = \left\{\nu^{-1}(\xi-\xi_0),\ (\nu T(\lambda,\nu))^{-1}(x-x_0-2t\xi)\right\}$$

which commute with $i\partial_t - \Delta$. The following lemma shows that u_{x_0,ξ_0} is concentrated in a tube

$$T^\nu_{x_0,\xi_0} = \{(x,\xi) : |x-x_0-2t\xi_0|\le \nu T(\lambda,\nu), |\xi-\xi_0|\le\nu\}$$

and decays rapidly away from it.

Lemma 56. *The solutions u_{x_0,ξ_0} for the* (124) *flow satisfy*

(138)
$$\sum_{(x_0,\xi_0)\in\mathbb{Z}_\nu^{2\times 3}}\sum_{|\alpha|\le N}\|L^\alpha_{x_0,\xi_0}u_{x_0,\xi_0}(t)\|^2_{U^2_{A,\nu,\lambda}(I,L^2)} \lesssim_A \|u_0\|^2_{L^2}, \qquad |I|\le T(\lambda,\nu)$$

Proof. At time 0 we clearly have

$$\sum_{|\alpha|\le N}\|L^\alpha_{x_0,\xi_0}u_{x_0,\xi_0}(0)\|^2_{L^2}\lesssim\|u_0\|^2_{L^2}$$

therefore it suffices to prove that a single generalized wave packet satisfies

$$\sum_{|\alpha|\le N}\|L^\alpha_{x_0,\xi_0}u_{x_0,\xi_0}\|^2_{U^2_{A,\nu}(I,L^2)}\lesssim_A \sum_{|\alpha|\le N}\|L^\alpha_{x_0,\xi_0}u_{x_0,\xi_0}(0)\|^2_{L^2} \tag{139}$$

This follows iteratively from

$$\|L_{x_0,\xi_0}v\|^2_{U^2_{A,\nu,\lambda}(I,L^2)}\lesssim_A \|L_{x_0,\xi_0}v(0)\|^2_{L^2}+\|v\|^2_{U^2_{A,\nu.\lambda}(I,L^2)} \tag{140}$$

for which, in turn, we need the commutator bound

$$\|[L_{x_0,\xi_0},A_{<\nu}\tilde S_\lambda\nabla]v\|_{DU^2_{A,\nu,\lambda}(I,L^2)}\lesssim_A\|v\|^2_{U^2_{A,\nu,\lambda}(I,L^2)} \tag{141}$$

as well as the similar one for the operator $\tilde S_\lambda A_{<\nu}\nabla$.

If $L_{x_0,\xi_0}=\nu^{-1}(\xi-\xi_0)$ then

$$[L_{x_0,\xi_0},A_{<\nu}\tilde S_\lambda\nabla]=\nu^{-1}(\nabla A_{<\nu})\tilde S_\lambda\nabla$$

therefore by duality we need to show that

$$\left|\int_I\int_{\mathbb{R}^3}\nabla A_{<\nu}\nabla\tilde S_\lambda u\bar v\,dxdt\right|\lesssim_A \nu\|A\|_{U^2_AH^1}\|u\|_{U^2_{A,\nu,\lambda}}\|v\|_{V^2_{A,\nu,\lambda}}$$

which follows from (135).

If $L_{x_0,\xi_0} = (\nu T(\lambda,\nu))^{-1}(x - x_0 - 2t\xi)$ then

$$[L_{x_0,\xi_0}, A_{<\nu}\tilde{S}_\lambda \nabla] = (\nu T(\lambda,\nu))^{-1}(A_{<\nu} + t(\nabla A_{<\nu})\nabla)\tilde{S}_\lambda$$

The second term is as above. For the first by duality we need to show that

$$\left| \int_I \int_{\mathbb{R}^3} A_{<\nu}\tilde{S}_\lambda u\bar{v}dxdt \right| \lesssim_A \nu T(\lambda,\nu)\|A\|_{U^2_A H^1}\|u\|_{U^2_{A,\nu,\lambda}}\|v\|_{V^2_{A,\nu,\lambda}}$$

which is much weaker and follows again from (135).

□

The parameter ν is chosen so that the packets move away from their initial support by the time $\lambda^{-\alpha}$. For this we impose the condition

$$\nu T(\lambda,\nu) < \lambda^{1-\alpha-\epsilon}$$

for some $\epsilon > 0$. This is satisfied if we choose ν of the form

$$\nu = \lambda^{1-\frac{\alpha}{4}-\epsilon}$$

in which case we have

$$\nu T(\lambda,\nu) = \lambda^{-\frac{3\alpha}{4}-2\varepsilon}(\log\lambda)^{-1}$$

This leads to the choice of β in the proposition. Indeed, the proof of the proposition is concluded due to the following lemma:

Lemma 57. *Let $\varepsilon > 0$. Choose ν so that $\nu T(\lambda,\nu) < \lambda^{1-\alpha-\varepsilon}$ and $\nu < \lambda^{1-\varepsilon}$. Then* (131)-(134) *hold for $|I| < T(\lambda,\nu)$.*

Proof. By the previous lemma we can replace the $(A_{\ll\lambda},\lambda)$ flow by the $(A_{<\nu},\lambda)$ flow in (131)-(134). We begin with (131). Without any restriction in generality we can assume that u and v are $U^2_{A,\nu,\lambda}L^2$ atoms. The generalized wave packet decomposition in Lemma 56 easily extends to $U^2_{A,\nu,\lambda}L^2$ atoms. Indeed, if we denote by $S_\nu(t,s)$ the evolution generated by (124) then an atom u_λ of the form

$$u_\lambda(t) = \sum_k 1_{[t_k,t_{k+1})}S_\nu(t,t_k)u^k_\lambda$$

can be partitioned as

$$u = \sum_{x_0}\sum_{|\xi_0|\approx\lambda} u_{x_0,\xi_0} \tag{142}$$

where

$$u_{x_0,\xi_0}(t) = \sum_k 1_{[t_k,t_{k+1})}S_\nu(t,t_k)u^k_{x_0,\xi_0} \tag{143}$$

with

$$u^k_{x_0,\xi_0} = \phi^\nu_{x_0+2t_k\xi_0,\xi_0}(x,D)u^k_\lambda$$

Then, denoting

$$|\!|\!| u_{x_0,\xi_0} |\!|\!|^2 = \sum_k \sum_{|\alpha|\le N} \|L^\alpha_{x_0,\xi_0}(t_k)u^k_{x_0,\xi_0}\|^2_{L^2}$$

we have the orthogonality relation

$$\sum_{x_0} \sum_{|\xi_0| \approx \lambda} \|u_{x_0,\xi_0}\|^2 \lesssim \sum_k \|u_k\|_{L^2}^2 \tag{144}$$

We argue in a similar manner for v. Hence it suffices to take u_λ as in (142),(143), and similarly for v_λ, and prove that for $|I| \leq T(\lambda,\nu)$ we have

$$\begin{aligned} \left| \int_I \int_{\mathbb{R}^3} B_\mu u_\lambda \bar{v}_\lambda dx dt \right| \lesssim_A & \; RHS(131) \cdot \|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{x_0} \sum_{|\xi_0|\approx\lambda} \|u_{x_0,\xi_0}\|^2 \right)^{\frac12} \left(\sum_{x_0} \sum_{|\xi_0|\approx\lambda} \|v_{x_0,\xi_0}\|^2 \right)^{\frac12} \end{aligned} \tag{145}$$

We proceed with several reductions, which will eventually lead to shorter time intervals.

1. Reduction to spatial scale $\lambda T(\lambda,\nu)$. Heuristically, in time $T(\lambda,\nu)$ the frequency λ waves for (124) travel by $\lambda T(\lambda,\nu)$. Hence we partition the space into cubes $\{Q_j\}_{j\in\mathbb{Z}^3}$ of size $\lambda T(\lambda,\nu)$. Correspondingly, we decompose u_λ into

$$u_\lambda = \sum_{j\in\mathbb{Z}^3} u_j, \qquad u_j = \sum_{x_0 \in Q_j} u_{x_0,\xi_0}$$

and similarly for v. By (138) the functions u_j decay rapidly away from an enlargement of Q_j. Precisely, if $x_0 \in Q_j$ and $|j-k| \geq 10$ then the separation between the tube $T^\nu_{x_0,\xi_0}$ and Q_k is $O(|j-k|\lambda T(\lambda,\nu))$. Comparing this with the tube thickness $\nu T(\lambda,\nu))$ we have

$$|u_j(t,x)| \lesssim \lambda^{-N} |j-k|^{-N} \sum_{x_0\in Q_j} \sum_{|\xi_0|\approx\lambda} \|u_{x_0,\xi_0}\|^2, \quad x \in Q_k,\ |j-k| \geq 10$$

Thus in (145) it suffices to consider the output of u_j and v_{j_1} for $|j-j_1| < 20$, and only within an enlargement CQ_j of Q_j; the rest is trivially estimated using the above bound. Furthermore, by Cauchy-Schwartz it suffices to consider a fixed j and k. By a slight abuse of notation we set $k = j$ in the sequel. Then (145) is reduced to

$$\begin{aligned} \left| \int_I \int_{\mathbb{R}^3} \chi_{CQ_j} B_\mu u_j \bar{v}_j dx dt \right| \lesssim_A & \; RHS(131) \cdot \|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{x_0\in Q_j} \sum_{|\xi_0|\approx\lambda} \|u_{x_0,\xi_0}\|^2 \right)^{\frac12} \left(\sum_{x_0\in Q_j} \sum_{|\xi_0|\approx\lambda} \|v_{x_0,\xi_0}\|^2 \right)^{\frac12} \end{aligned} \tag{146}$$

2. Reduction to small angles. Here we partition the λ annulus A_λ in frequency into small angles of size $\frac{1}{10}$ with centers in $\Theta \subset \mathbb{S}^2$,

$$A_\lambda = \bigcup_{\theta\in\Theta} A_{\lambda,\theta}$$

Then we divide

$$u_j = \sum_{\theta\in\Theta} u_{j,\theta}, \qquad u_{j,\theta} = \sum_{x_0\in Q_j}\sum_{\xi\in A_\theta} u_{x_0,\xi_0}$$

and similarly for v_j. It remains to prove that

$$\begin{aligned} \left|\int_I\int_{\mathbb{R}^3} \chi_{CQ_j} B_\mu u_{j,\theta}\bar v_{j,\omega}dxdt\right| \lesssim_A & \ RHS(131)\cdot \|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{x_0\in Q_j}\sum_{\xi_0\in A_{\lambda,\theta}} \|u_{x_0,\xi_0}\|^2\right)^{\frac12}\left(\sum_{x_0\in Q_j}\sum_{\xi_0\in A_{\lambda,\omega}} \|v_{x_0,\xi_0}\|^2\right)^{\frac12} \end{aligned} \tag{147}$$

3. Reduction to a spatial strip of size $\lambda^\varepsilon\nu T(\lambda,\nu)$. Given directions θ and ω as above we choose a coordinate, say ξ_1, so that both the θ and the ω sectors A_θ, A_ω are away from $\xi_1 = 0$. Dividing the spatial coordinates $x = (x_1, x')$ we partition the space into strips S_k of thickness $\lambda^\varepsilon\nu T(\lambda,\nu)$ in the x_1 direction.

Arguing as in (13), the A factor is square summable with respect to the strips S_k. There are about $\lambda^{1-\epsilon}\nu^{-1}$ such strips which intersect $30Q_j$. Hence by losing a $\lambda^{\frac{1-\varepsilon}{2}}\mu^{-\frac12}$ factor we can use Holder's inequality to reduce the problem to the case when A is supported in a single spatial strip S_k. It remains to prove that

$$\begin{aligned} \left|\int_I\int_{\mathbb{R}^3} \chi_{S_k\cap CQ_j} B_\mu u_{j,\theta}\bar v_{j,\omega}dxdt\right| \lesssim_A & \ RHS(131)\cdot \lambda^{-\frac{1-\varepsilon}{2}}\mu^{\frac12}\|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{x_0\in Q_j}\sum_{\xi_0\in A_{\lambda,\theta}} \|u_{x_0,\xi_0}\|^2\right)^{\frac12}\left(\sum_{x_0\in Q_j}\sum_{\xi_0\in A_{\lambda,\omega}} \|v_{x_0,\xi_0}\|^2\right)^{\frac12} \end{aligned} \tag{148}$$

4. Reduction to spatial scale $\lambda^\varepsilon\nu T(\lambda,\nu)$. Due to the choice of coordinates above, the packets in $u_{j,\alpha}$ and $v_{j,\omega}$ travel in directions which are transversal to S_k. Hence if we partition S_k into cubes $\tilde Q_l$ of size $\lambda^\varepsilon\nu T(\lambda,\nu)$, each packet will intersect only finitely many cubes. Then we partition further

$$u_{j,\theta} = \sum_l u_{j,\theta,l}, \qquad u_{j,\theta,l} = \sum_{(x_0,\xi_0)\in A_{j,\theta,l}} u_{x_0,\xi_0}$$

where

$$A_{j,\theta,l} = \{(x_0,\xi_0); x_0\in Q_j,\ \xi_0\in A_{\lambda,\theta},\ \{x_0+\mathbb{R}\xi_0\}\cap\tilde Q_l \neq \emptyset\}$$

and packets u_{x_0,ξ_0} intersecting more than one cube are arbitrarily placed in one of the terms.

Arguing as in the first reduction, for $|l-l_1|\gg 1$ the size of $u_{j,\theta,l}$ in $\tilde Q_{l_1}$ is rapidly decreasing,

$$|u_{j,\theta,l}(t,x)|^2 \lesssim \lambda^{-N}|l-l_1|^{-N}\sum_{(x_0,\xi_0)\in A_{j,\theta,l}} \|u_{x_0,\xi_0}\|^2, \quad x\in\tilde Q_{l_1},\ |l-l_1|\gg 1$$

Hence the joint contribution of $u_{j,\theta,l}$ and v_{j,ω,l_1} in (148) is nontrivial only on the diagonal $|l-l_1| \lesssim 1$. By Cauchy-Schwartz with respect to l it suffices to estimate this contribution for fixed l, l_1, in an enlarged cube $C\tilde{Q}_l$. By a slight abuse of notation we set $l = l_1$. Then we need to show that

$$\begin{aligned} \left| \int_I \int_{\mathbb{R}^3} \chi_{CQ_l} B_\mu u_{j,\theta,l} \bar{v}_{j,\omega,l} dxdt \right| \lesssim_A & RHS(131) \cdot \lambda^{-\frac{1-\varepsilon}{2}} \mu^{\frac{1}{2}} \|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{(x_0,\xi_0) \in A_{j,\theta,l}} \|u_{x_0,\xi_0}\|^2 \right)^{\frac{1}{2}} \left(\sum_{(x_0,\xi_0) \in A_{j,\omega,l}} \|v_{x_0,\xi_0}\|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{149}$$

5. Reduction to a time interval of size $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$. Each tube T_{x_0,ξ_0} intersects the cube Q_l in a time interval of size $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$. Hence it is natural to partition the time interval I into subintervals I_m of length $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$. Correspondingly, we split $u_{j,\theta,l}$ into

$$u_{j,\theta,l} = \sum_m u_{j,\theta,l,m}, \qquad u_{j,\theta,l,m} = \sum_{(x_0,\xi_0)\in A_{j,\theta,l,m}} u_{x_0,\xi_0}$$

where

$$A_{j,\theta,l,m} = \{(x_0,\xi_0); x_0 \in Q_j,\ \xi_0 \in A_{\lambda,\theta},\ T^\nu_{x_0,\xi_0} \cap I_m \times \tilde{Q}_l \neq \emptyset\}$$

and similarly for $v_{j,\omega,l}$. Again, the size of $u_{j,\theta,l,m}$ in $I_{m_1} \times Q_l$ is negligible if $|m-m_1| \gg 1$. Thus by Cauchy-Schwartz with respect to m the estimate (149) reduces to the case of a single interval I_m,

$$\begin{aligned} \left| \int_{I_m} \int_{\mathbb{R}^3} \chi_{CQ_l} B_\mu u_{j,\theta,l,m} \bar{v}_{j,\omega,l,m} dxdt \right| \lesssim_A & RHS(131) \lambda^{-\frac{1-\varepsilon}{2}} \mu^{\frac{1}{2}} \|B_\mu\|_{U^2_W L^2} \\ & \left(\sum_{(x_0,\xi_0) \in A_{j,\theta,l,m}} \|u_{x_0,\xi_0}\|^2 \right)^{\frac{1}{2}} \left(\sum_{(x_0,\xi_0) \in A_{j,\omega,l,m}} \|v_{x_0,\xi_0}\|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{150}$$

But this follows from the hypothesis since

$$|I_m| = \lambda^{\varepsilon-1}\nu T(\lambda,\nu) \leq \lambda^{-\alpha}$$

and

$$|I|^{\frac{1}{2}} \lambda^{-\frac{1-\varepsilon}{2}} \mu^{\frac{1}{2}} = |I_m|$$

In the case of (132) the argument is similar but with several adjustments which we outline.

1. Reduction to spatial scale $\max\{\mu, \lambda^\varepsilon \nu\} T(\lambda,\nu)$. This smaller initial localization scale is possible since frequency ν Schrödinger waves travel with speed μ, so within time $T(\lambda,\nu)$ they can spread only as far as $\mu T(\lambda,\nu)$. Thus for the u_μ factor we have square summability on the $\mu T(\lambda,\nu)$. For the wave factor B_λ by (13) we have square summability on the same scale, therefore we are allowed to localize spatially the estimate on the $\mu T(\lambda,\nu)$ scale. If μ is small we only can take partial advantage of this due to the wider spread of frequency λ pakets.

2. Reduction to small angles. This is as before.

3. Reduction to a spatial strip of size $\lambda^\varepsilon \nu T(\lambda,\nu)$. The difference here is that we have only about $\max\{1, \mu\lambda^{-\varepsilon}\nu^{-1}\}$ strips intersecting a $\max\{\mu, \lambda^\varepsilon\nu\}T(\lambda,\nu)$ cube, therefore we only loose a factor of

$$\max\{1, \mu^{\frac{1}{2}}\lambda^{-\frac{\varepsilon}{2}}\nu^{-\frac{1}{2}}\}$$

4. Reduction to spatial scale $\lambda^\varepsilon \nu T(\lambda,\nu)$. This is as before.

5. Reduction to a time interval of size $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$. As before, the frequency λ wave packets spend a time $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$ inside a $\lambda^\varepsilon\nu T(\lambda,\nu)$ cube Q. However, the frequency μ packets spend a longer time $\mu^{-1}\lambda^\varepsilon\nu T(\lambda,\nu)$ inside Q. Hence we can carry out first a lossless reduction down to time scale $\min\{1, \mu^{-1}\lambda^\varepsilon\nu\}T(\lambda,\nu)$. To further reduce the time scale to $\lambda^{\varepsilon-1}\nu T(\lambda,\nu)$ we can only use the square summability for the frequency λ waves, therefore we apply Cauchy-Schwartz and loose an additional factor of

$$\min\{1, \mu^{-1}\lambda^\varepsilon\nu\}^{\frac{1}{2}}(\lambda^{\varepsilon-1}\nu)^{-\frac{1}{2}}$$

Finally, combining the two losses in Steps 3 and 5 we obtain a total loss of

$$(\lambda^{\varepsilon-1}\nu)^{-\frac{1}{2}}$$

which is identical to the one in Step 3 of the proof of (131). We conclude as before.

The argument is considerably simpler in the case of (133) and (134). There each packet intersects $Q \times I$ in a time interval which is shorter than $\lambda^{-1}\nu T(\lambda,\nu) < \lambda^{-\alpha}$. Grouping the wave packets with respect to such time intervals we obtain the square summability of the outputs and reduce the problem to the shorter time scale $\lambda^{-\alpha}$.

□

□

References

[1] Ioan Bejenaru and Daniel Tataru. Large data local solutions for the derivative nls equation. To appear, JEMS, arXiv:math/0610092.

[2] Yan Guo, Kuniaki Nakamitsu, and Walter Strauss. Global finite-energy solutions of the Maxwell-Schrödinger system. *Comm. Math. Phys.*, 170(1):181–196, 1995.

[3] Martin Hadac, Sebastian Herr, and Herbert Koch. Well-posedness and scattering for the kp-ii equation in a critical space. arXiv:0708.2011.

[4] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[5] Herbert Koch, Hart F. Smith, and Daniel Tataru. Subcritical l^p bounds on spectral clusters for lipschitz metrics. Preprint.

[6] Herbert Koch and Daniel Tataru. A-priori bounds for the 1-d cubic nls in negative sobolev spaces. To appear, IMRN, arXiv:math/0612717.

[7] Herbert Koch and Daniel Tataru. Dispersive estimates for principally normal pseudodifferential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.

[8] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Wave packet parametrices for evolutions governed by pdo's with rough symbols. To appear, Proc. AMS, arXiv:math/0611252.

[9] Makoto Nakamura and Takeshi Wada. Local well-posedness for the Maxwell-Schrödinger equation. *Math. Ann.*, 332(3):565–604, 2005.

[10] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with non-smooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.

[11] Daniel Tataru. Phase space transforms and microlocal analysis. In *Phase space analysis of partial differential equations. Vol. II*, Pubbl. Cent. Ric. Mat. Ennio Giorgi, pages 505–524. Scuola Norm. Sup., Pisa, 2004.

[12] Yoshio Tsutsumi. Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions. *Comm. Math. Phys.*, 151(3):543–576, 1993.

Department of Mathematics, Texas A&M University

Department of Mathematics, University of California, Berkeley